Retarded Green Functions

- Our goal is to write down the retarded Green function of the Maxwell equation and to learn mathematics.

- Let us start with the harmonic oscillator

$$\underbrace{\left[\frac{m\,d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2\right]}_{\equiv \mathcal{L}_t} G_R(t,t_0) = \delta(t-t_0)$$

$G_R(t,t_0)$ is the displacement at time t, due to an impulsive force at time t_0. Here we have defined the linear operator $\mathcal{L}_t$ which is the underlined term. For a general $F(t)$ driving the oscillator

$$\mathcal{L}_t\, x(t) = F(t)$$

The general solution is a specific solution $X_s(t)$ (usually the steady state) + a homogeneous solution $X_{homo}(t)$

$$X(t) = X_s(t) + \chi_{homo}(t)$$

Where

$$\mathcal{L}_t X_s(t) = F(t) \quad \text{and} \quad \mathcal{L}_t \chi_{homo}(t) = 0$$

and χ_{homo} is adjusted to satisfy the initial conditions

For the oscillator example at small damping

$$X_{homo}(t) = A e^{-\eta/2\, t} e^{-i\omega_0 t} + B e^{-\eta/2\, t} e^{i\omega_0 t}$$

The specific solution

$$(1) \qquad X_s(t) = \int_{-\infty}^{\infty} G_R(t-t_0)\, F(t_0)\, dt_0$$

The homogeneous solution will decay away in time leaving the specific solution. This clearly satisfies the equation

$$\mathcal{L}_t X_s(t) = \int_{-\infty}^{\infty} \mathcal{L}_t G(t,t_0)\, F(t_0)\, dt_0$$

$$= \int_{-\infty}^{\infty} \delta(t-t_0)\, F(t_0)\, dt_0 = F(t)$$

- We will specifically be interested in the retarded or causal Green fcn:

$$G_R(t,t_0) = 0 \quad \text{for} \quad t < t_0$$

So G_R is a response at t to a force at t_0

Note all physical quantities are ultimately expressible as Grn-fcns. For example, we used a harmonic oscillator (Lorentz Model) to describe the dielectric constant. $\vec{F}(t) = eE(t)$ the current $\vec{j}(t) = ne\,v(t)$, so from Eq. (1)

$$x_s(\omega) = G_R(\omega)\, F(\omega)$$

And

$$j(\omega) = ne\,\underbrace{(-i\omega\, x(\omega))}_{v(\omega)}$$

$$= ne^2\, G_R(\omega)\,(-i\omega E(\omega))$$

$F(\omega) = e\,E(\omega)$

So comparison with the constitutive relation $(j(\omega) = \chi_e(\omega)\,(-i\omega E(\omega)))$ gives

$$\chi_e(\omega) = ne^2 G_R(\omega)$$

Thus we see how, in a particular model, the response function of the dynamical system determines the susceptibility

Finding the Green Function in time : Direct Method

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2 \right] G_R(t,t_0) = \delta(t-t_0)$$

Demand continuity and integrate from $t_0-\varepsilon$ to $t_0+\varepsilon$. We know $G_R(t,t_0)=0$ for $t<t_0$

☆ $$G_R(t_0+\varepsilon, t_0) = 0$$

Then we have

$$m\frac{d}{dt}G_R(t_0+\varepsilon,t_0) + m\eta\, G_R(t_0+\varepsilon,t_0)\Big|_{\text{lm } \varepsilon\to 0} = 1$$

Or

☆☆ $$m\frac{d}{dt}G_R(t_0+\varepsilon,t_0) = 1$$

Then we can solve the diff-eq given the initial conditions. The two homogeneous solutions are

$$x_\pm = e^{-\eta/t}\, e^{\pm i\omega_0 t} \quad \text{for small } \eta$$

Then the linear combo of $x_\pm$ which satisfies the initial conditions (☆) and (☆☆) are

$$G_R = \begin{cases} \sin\omega_0(t-t_0)\, e^{-\eta/2(t-t_0)}/m\omega_0 & t-t_0>0 \\ 0 & \text{otherwise} \end{cases}$$

Usually this is written

$$G_R(t) = \Theta(\tau) \frac{\sin \omega_0 \tau}{m\omega_0} e^{-\frac{\gamma}{2}\tau} \qquad \tau \equiv t - t_0$$

Fourier Method for Green fcn

$$\int \frac{d\omega}{2\pi} e^{-i\omega\tau}$$

$$\left[m\frac{d^2}{d\tau^2} + m\eta\frac{d}{d\tau} + m\omega_0^2 \right] G_R(\tau) = \delta(\tau)$$

Fourier Transform both sides

$$\left[-m\omega^2 + m\eta(-i\omega) + m\omega_0^2 \right] G_R(\omega) = 1$$

$$G_R(\omega) = \frac{1/m}{\left[-\omega^2 + \omega_0^2 - i\omega\eta \right]}$$

Thus

$$G_R(\tau) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}\, 1/m}{\left[-\omega^2 + \omega_0^2 - i\omega\eta \right]}$$

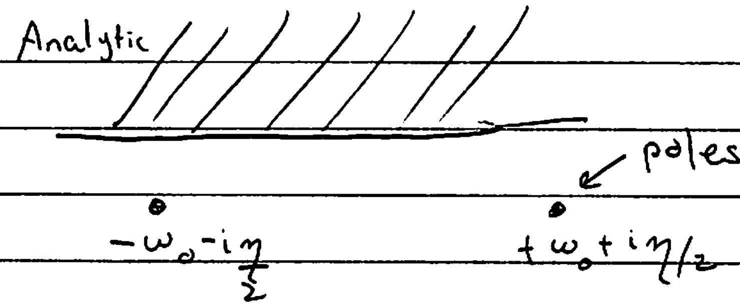
You can do these integrals with contour integration
the poles are at

$$\omega^2 + i\omega\eta = \omega_0^2$$

Solving this equation for small η:

$$\omega \simeq \pm\omega_0 - i\frac{\eta}{2}$$

We see that the integrand has the following analytic structure



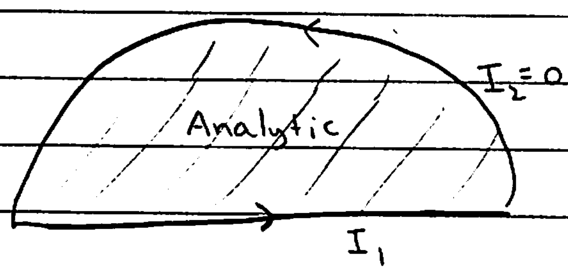
So now we should do the integral:

Case 1: $\tau < 0$ $\quad G_R(\tau) = 0 \leftarrow$ causality

The math works like this, since $\tau < 0$:

$$e^{-i\omega\tau} \xrightarrow[\omega \to \text{complex}]{} e^{-i\,\mathrm{Re}\,\omega\,\tau}\, \underbrace{e^{+[\mathrm{Im}\,\omega]\tau}}_{\text{decreasing exponentially for } \mathrm{Im}\,\omega > 0} \quad \tau < 0$$

Thus for $\tau < 0$ we can close the contour in the UHP without picking up poles and find zero



$$I_1 + \cancel{I_2}^{\,0} = 0$$
$$I_1 = 0$$

Case 2: $\tau > 0$

For $\tau > 0$ we must close the contour in the LHP picking up poles at $\omega = \pm\omega_0 - i\frac{\eta}{2}$

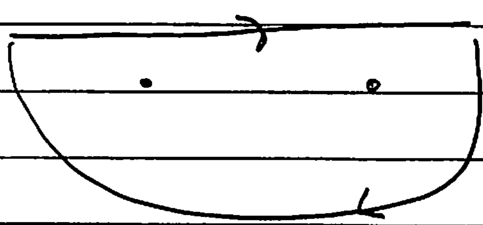
For $\tau > 0$: wrong way around poles

$$G_R(\tau) = -2\pi i \left[\mathrm{Res}_{\omega = \omega_0 - i\frac{\eta}{2}} + \mathrm{Res}_{\omega = -\omega_0 - i\eta/2} \right]$$

$$= \frac{1}{m}\frac{i}{2\omega_0} e^{-\eta/2 t} e^{-i\omega_0 \tau} + \frac{1}{m}\frac{-i}{2\omega_0} e^{-\eta/2 \tau} e^{+i\omega_0 \tau}$$

homogeneous solutions

from two pages back



$$= \frac{1}{m} e^{-\eta/2 \tau} \frac{\sin\omega_0 \tau}{\omega_0}$$

So

$$G_R(\tau) = \Theta(\tau) \frac{\sin\omega_0\tau}{m\omega_0} e^{-\eta/2 \tau} \xrightarrow[\eta \to 0]{} \Theta(\tau) \frac{\sin\omega_0 \tau}{m\omega_0}$$

We will see that this Green function is closely related to the green function of the wave eqn

(Aside: $i\varepsilon$ perscription :)

Take the $\eta \to 0$ limit of the damped harmonic oscillator

$$G_R(\tau) = \frac{\sin \omega_0 \tau}{m\omega_0}\,\Theta(\tau)$$

$$G_R(\omega) = \frac{1/m}{[-\omega^2 + \omega_0^2]}$$

But this is ambiguous since the poles are on the real axis. What does this mean $\int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{(-\omega^2+\omega_0^2)}$?

We know that causality demands that the poles lie in the lower half plane. We can enforce this by adding an infinitessimal imaginary part

$$\omega \to \omega + i\varepsilon \leftarrow \text{positive}$$

So

$$G_R(\omega) = \frac{1/m}{(-(\omega+i\varepsilon)^2 + \omega_0^2)}$$

$$\simeq \frac{1/m}{(-\omega^2 + \omega_0^2 - 2i\varepsilon\omega)}$$

Amounts to adding an infinitessimal damping coeffici[ent]

$\eta = 2\varepsilon$

Kramers - Krönig and retarded Green functions

The Kramers Krönig relation hold for causal response functions, which are always analytic in UHP (upper half plane). $G_R(\omega)$ satisfies these properties, thus:

$$\mathrm{Re}\, G_R(\omega) = - \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \cdot \frac{P}{\omega - \omega'} \, \mathrm{Im}\, G_R(\omega')$$

$$\mathrm{Im}\, G_R(\omega) = + \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \, \frac{P}{\omega - \omega'} \, \mathrm{Re}\, G_R(\omega')$$