Problem 1. Retarded time derivatives

(a) Compute $\partial T/\partial t$ and $\partial T/\partial \boldsymbol{r}^i$. What is

$$\frac{\partial T}{\partial t} + c\boldsymbol{n}^i \frac{\partial T}{\partial \boldsymbol{r}^i} \,. \tag{1}$$

Explain physically.

Problem 2. Lienard-Wiechert for constant velocity

For a particle moving with constant velocity v along the $x-$axis you showed previously using Lorentz transformation that

$$A^x(t, x, \boldsymbol{x}_\perp = \boldsymbol{b}) = \frac{e}{4\pi} \frac{\gamma\beta}{\sqrt{b^2 + \gamma^2(x - vt)^2}} \tag{2}$$

so that at the observation point $(t, \boldsymbol{r}) = (t, x, y, z) = (t, 0, b, 0)$ the potenial is

$$A^x(t, x, y = b) = \frac{e}{4\pi} \frac{\gamma\beta}{\sqrt{b^2 + (\gamma vt)^2}} \tag{3}$$

Start by noting the definitions

$$T \equiv t - \frac{R}{c} \qquad R = |\boldsymbol{r} - \boldsymbol{r}_*(T)| \qquad \boldsymbol{R} \equiv R\boldsymbol{n} \equiv \boldsymbol{r} - \boldsymbol{r}_*(T) \qquad \boldsymbol{n} \equiv \frac{\boldsymbol{R}}{R} \tag{4}$$

and drawing a picture for yourself.

(a) Show that the Lienard Wiechert result,

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e}{4\pi} \left[\frac{\mathbf{v}/c}{R(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \right]_{\rm ret} . \tag{5}$$

gives the same result as Eq. (3).

(b) Show that the Lienard-Wiechert potential, Eq. (5), and analogous equation for φ can be written covariantly

$$A^\mu(X) = -\frac{e}{4\pi} \left[\frac{U^\mu}{U \cdot \Delta X} \right]_{\rm ret} , \tag{6}$$

where ΔX^μ is the difference in the space-time coordinate four vectors of the emission and observation points, and U^μ is the four velocity of the particle. What is $\Delta X \cdot \Delta X \equiv \Delta X^\mu \Delta X_\mu$?

Problem 3. The Hamiltonian of a Relativistic Particle

In class we discussed the point particle Lagrangian

$$L = -mc^2\sqrt{1 - \dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}}/c^2} - e\varphi + \frac{e}{c}\,\dot{\boldsymbol{x}} \cdot \boldsymbol{A}\,. \tag{7}$$

(a) Show that the canonical momentum is

$$\boldsymbol{p} = \boldsymbol{p}_{\rm kin} + \frac{e}{c}\boldsymbol{A}\,, \tag{8}$$

where the kinetic momentum is $\boldsymbol{p}_{\rm kin} = \gamma m \dot{\boldsymbol{x}}$.

(b) Show that the Hamiltonian is

$$H = c\sqrt{(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2 + (mc)^2} + e\varphi\,. \tag{9}$$

It is the canonical momentum which appears in the Hamiltonian, but the kinetic momentum which appears in

$$\frac{d\boldsymbol{p}_{\rm kin}}{dt} = q(\boldsymbol{E} + \frac{\mathbf{v}}{c} \times \boldsymbol{B})\,. \tag{10}$$

(c) What is the Hamiltonian in the non-relativistic limit?

Problem 4. (Optional) Variational derivatives for students

- Variational derivatives cause students great hardship. Its meaning is discussed in what follows. We are considering an integral[1] depending on a path $x(t)$ starting at x_1 and ending at x_2. For example

$$I[x] = \int_{t_1,x_1}^{t_2,x_2} dt\, L(x(t))\,. \tag{11}$$

Then we deform the path

$$x(t) \to x(t) + \delta x(t) \tag{12}$$

where the endpoints are unchanged $\delta x(t_1) = \delta x(t_2) = 0$. Then the integral changes and the result must be proportional to $\delta x(t)$ for smal variations

$$\delta I[x] = \int dt \left[\frac{\partial L(x(t))}{\partial x(t)} \right] \delta x(t) \tag{13}$$

We say that the thing in square bracekts (i.e. the thing sitting in front of $\int dt\, \delta x(t)$) is the variation derivative of the functional

$$\frac{\delta I[x]}{\delta x(t)} = \text{thing in front of } \textstyle\int dt\, \delta x(t) \quad = \frac{\partial L(x(t))}{\partial x(t)} \tag{14}$$

When working with variations, I prefer to work with the change in the integral (i.e. Eq. (13)), which somehow means more to me than some mysterious new differentiation symbol, and always works.

- However, as the formalism of variational derivatives is common, let us develop it. Clearly

$$x(t) = \int dt\, x(t')\, \delta(t - t')\,. \tag{15}$$

Then following the steps leading to Eq. (13) and Eq. (14) we see that

$$\frac{\delta x(t)}{\delta x(t')} = \delta(t - t')\,. \tag{16}$$

Then the normal rules of differentiation apply

$$\frac{\delta L(x(t'))}{\delta x(t)} \equiv \frac{\partial L(x(t'))}{\partial x(t')} \frac{\delta x(t')}{\delta x(t)} = \frac{\partial L(x(t'))}{\partial x(t')} \delta(t' - t)\,. \tag{17}$$

In this way if

$$I[x] = \int_{t_1,x_1}^{t_2,x_2} dt'\, L(x(t'))\,, \tag{18}$$

[1] Technically the integral is a functional of $x(t)$, *i.e.* something which takes a function $(x(t))$ and spits out a number.

then we can differentiate under the integral

$$\frac{\delta I[x]}{\delta x(t)} = \int_{t_1,x_1}^{t_2,x_2} dt' \, \frac{\delta L(x(t'))}{\delta x(t)} \, , \tag{19}$$

$$= \int_{t_1,x_1}^{t_2,x_2} dt' \, \frac{\partial L(x(t'))}{\partial x(t)} \delta(t'-t) \tag{20}$$

$$= \frac{\partial L(x(t))}{\partial x(t)} \, , \tag{21}$$

as we got before

- Some people who do numerics like to work discretely where $x_i = x(t_i)$, with $t_i = t_1 + i\Delta t$ being discretely spaced points. Then the integral is an ordinary function of x_i

$$I(x_1, x_2, x_3 \ldots) = \sum_i \Delta t L(x_i) \tag{22}$$

Then the variational derivative is just limit as Δt goes to zero of

$$\frac{\delta I[x]}{\delta x(t_i)} = \frac{1}{\Delta t} \frac{\partial I}{\partial x_i} \tag{23}$$

- We have discussed a function of t and the integral which is a functional of $x(t)$. When working with fields which are a function of space-time $A(x)$ (here $x = (ct, \boldsymbol{x})$), the integral is functional of $A(x)$

$$I[A] = \int d^4x \, \mathcal{L}(A(x)) \, . \tag{24}$$

Then the variation of the integral is found by changing the function $A(x)$ to a new function

$$A(x) \to A(x) + \delta A(x) \, . \tag{25}$$

The integral then changes to $I \to I + \delta I$

$$\delta I = \int d^4x \left[\frac{\partial \mathcal{L}(A(x))}{\partial A(x)} \right] \delta A(x) \tag{26}$$

The thing in square brackets in front of $\int d^4x \, \delta A(x)$ is defined as the variational derivative

$$\frac{\delta I[A]}{\delta A(x)} = \text{thing in front of } \int d^4x \, \delta A(x) \tag{27}$$

$$= \frac{\partial \mathcal{L}(A(x))}{\partial A(x)} \text{in this simple case} \tag{28}$$

- In the same sense as before

$$A(x) = \int d^4y A(y) \delta^4(x-y) \,. \tag{29}$$

Thus

$$\frac{\delta A(x)}{\delta A(y)} = \delta^4(x-y) \,, \tag{30}$$

and

$$\frac{\delta \mathcal{L}(A(y))}{\delta A(x)} \equiv \frac{\partial \mathcal{L}(A(y))}{\partial A(y)} \delta^4(y-x) \,. \tag{31}$$

I have always found this slightly confusing and a bit too formal, and prefer the more understandable change in integral, Eq. (26).

We defined the current as the thing sitting in front of $\int d^4x \, \delta A_\mu(x)$ under a variation of the interaction lagrangian between the charge particles (or medium) and the fields, *i.e.*

$$\delta S_{\rm int} \equiv \int d^4x \frac{J^\mu(x)}{c} \delta A_\mu(x) \tag{32}$$

or

$$\frac{J^\mu(x)}{c} = \frac{\delta S_{\rm int}[A]}{\delta A_\mu(x)} \tag{33}$$

We also said the interation between a point particle and the field is

$$S_{\rm int-pp} = \frac{e}{c} \int d\tau \frac{dx_o^\mu(t)}{d\tau} A_\mu(x_o(\tau)) \tag{34}$$

where $x_o(\tau)$ is the trajectory of the particle.

(a) Show that for a point particle moving with trajectory $x_o^\mu(\tau)$, the current is $J^\mu(x)$ is

$$J^\mu_{\rm pp}(x) = \frac{e}{c} \int d\tau \frac{dx_o^\mu(\tau)}{d\tau} \delta^4(x - x_o(\tau)) \tag{35}$$

and how this reduces to

$$J^\mu(x) = e v^\mu \delta^3(\boldsymbol{x} - \boldsymbol{x}_o(t)) \,. \tag{36}$$

Note that $v^{mu} = (c, \mathbf{v})$ is not a four vector, although the current is not a four vector.

(b) (**Optional)** Show that

$$v^\mu \delta^3(\boldsymbol{x} - \boldsymbol{x}_o(t)) \,. \tag{37}$$

is a four vector

(c) Consider electrostatics, where $\boldsymbol{E}(t, \boldsymbol{x}) = -\nabla\varphi(\boldsymbol{x})$ and $\boldsymbol{B} = 0$. Starting from the action of electrodynamics

$$S = S_o + S_{\rm int} = \int d^4x \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{J^\mu}{c} A_\mu \,, \tag{38}$$

show that the action for the the electrostatic potential can be taken to be

$$S[\varphi(\boldsymbol{x})] = \int d^3\boldsymbol{x} \frac{1}{2}(\nabla\varphi(\boldsymbol{x}))^2 - \rho(\boldsymbol{x})\varphi(\boldsymbol{x})\,. \tag{39}$$

And show that a variation of the action gives the expected equation of motion for the electostatic potential.

(d) (**Optional**) Similarly, consider magnetostatics where $\boldsymbol{B}(t,\boldsymbol{x}) = \nabla\times\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{E} = 0$. Determine the action for the vector potential $\boldsymbol{A}(\boldsymbol{x})$ and vary this action to determine the equations of magnetostatics.