Problem 1. A conducting slab

A plane polarized electromagnetic wave $\boldsymbol{E} = \boldsymbol{E}_I e^{ikz-\omega t}$ is incident normally on a flat uniform sheet of an *excellent* conductor ($\sigma \gg \omega$) having thickness D. Assume that in space and in the conducting sheet $\mu = \epsilon = 1$, discuss the reflection an transmission of the incident wave.

(a) Show that the amplitudes of the reflected and transmitted waves, corrrect to first order in $(\omega/\sigma)^{1/2}$, are:

$$\frac{E_R}{E_I} = \frac{-(1-e^{-2\lambda})}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{1}$$

$$\frac{E_T}{E_I} = \frac{2\gamma e^{-\lambda}}{(1-e^{-2\lambda}) + \gamma(1+e^{-2\lambda})} \tag{2}$$

where

$$\gamma = \sqrt{\frac{2\omega}{\sigma}}(1-i) = \frac{\omega\delta}{c}(1-i) \tag{3}$$

$$\lambda = (1-i)D/\delta \tag{4}$$

and $\delta = \sqrt{2/\omega\mu\sigma}$ is the skin depth.

(b) Verify that for zero thickness and infinite skin depth you obtain the proper limiting results.

(c) **Optional:** Show that, except for sheets of very small thickness, the transmission coefficient is

$$T = \frac{8(\mathrm{Re}\gamma)^2 e^{-2D/\delta}}{1 - 2e^{-2D/\delta}\cos(2D/\delta) + e^{-4D/\delta}} \tag{5}$$

Sketch $\log T$ as a function of D/δ, assuming $\mathrm{Re}\gamma = 10^{-2}$. Define "very small thickness".

Problem 2. Exponentially Decaying Waves

Consider an exponentially decaying wave in vacuum moving in the $x - z$ plane

$$\boldsymbol{E}(t,x,z) = \boldsymbol{E}_o e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \boldsymbol{E}_o e^{ik_x x - \kappa_z z - i\omega t} \tag{6}$$

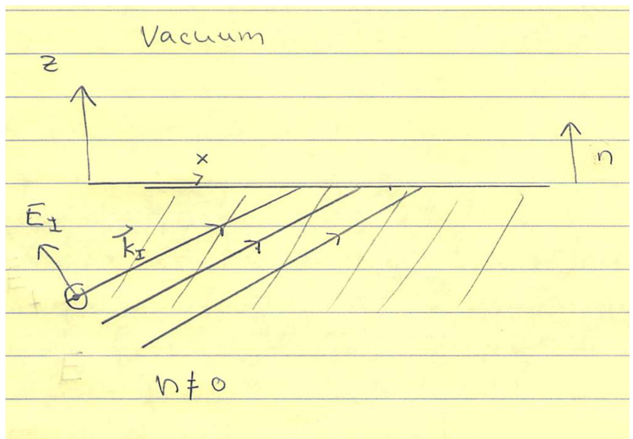
where $\boldsymbol{k} = (k_x, k_z) = (k_x, i\kappa_z)$, and $\boldsymbol{E}_o = (E_x, E_z)$ is polarized in the (x,z) plane, but is not necessarily real.

(a) Use Maxwell equations to determine the relation between k_x, κ_z and ω

(b) Show that the time averaged Poynting flux in the z direction $\boldsymbol{S}\cdot\hat{\boldsymbol{z}}$ is zero. (Hint: what are the constraints on $\boldsymbol{E}_o$ and $\boldsymbol{B}_o$ imposed by the Maxwell equations)

Problem 3. Analysis of the Good-Hänchen effect

A ribbon beam of in plane polarized radiation of wavelength λ is totally internally reflected at a plane boundary between a non-permeable (i.e. $\mu = 1$) dielectric media with index of refraction n and vacuum (see below). The critical angle for total internal reflection is θ_I^o, where $\sin\theta_I^o = 1/n$. First assume that the incident wave takes the form $\boldsymbol{E}(t, \boldsymbol{r}) = \boldsymbol{E}_I e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t}$ of a pure plane wave polarized in plane and study the transmitted and reflected waves.



(a) Starting from the Maxwell equations, show that for $z > 0$ (i.e. in vacuum) the electric field takes the form:

$$\boldsymbol{E}_2(x, z) = \boldsymbol{E}_2 e^{-\frac{\omega}{c}(\sqrt{n^2 \sin\theta_I^2 - 1})z} e^{i\frac{\omega n \sin\theta_I}{c} x} \tag{7}$$

(b) Starting from the Maxwell equations, show that for $\theta_I > \theta_I^0$ the ratio of the reflected amplitude to the incident amplitude is a pure phase

$$\frac{E_R}{E_I} = e^{i\phi(\theta_I, \theta_I^o)} \tag{8}$$

and determine the phase angle. Thus the reflection coefficient $R = |E_R/E_I|^2 = 1$ However, phase has consequences.

(c) Show that for a monochromatic (*i.e.* constant $\omega = ck$) ribbon beam of radiation in the z direction with a transverse electric field amplitude, $E(x)e^{ik_z z - i\omega t}$, where $E(x)$ is smooth and finite in the transverse extent (but many wavelengths broad), the lowest order approximation in terms of plane waves is

$$\boldsymbol{E}(x, z, t) = \boldsymbol{\epsilon} \int \frac{d\kappa}{(2\pi)} A(\kappa) e^{i\kappa x + ikz - i\omega t} \tag{9}$$

where $k = \omega/c$. Thus, the finite beam consists of a sum plane waves with a small range of angles of incidence, centered around the geometrical optics value.

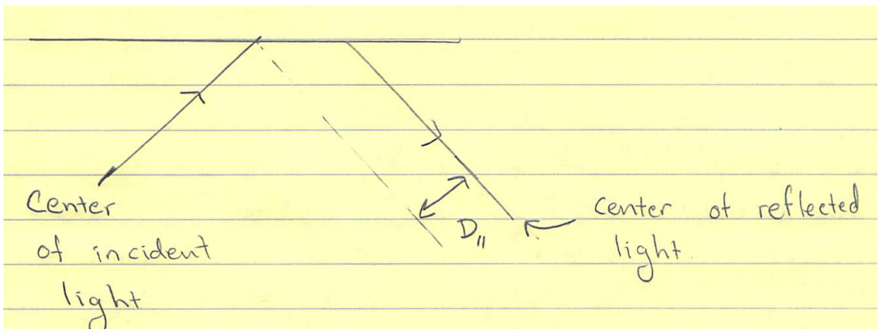
(d) Consider a reflected ribbon beam and show that for $\theta_I > \theta_I^o$ the electric field can be expressed approximately as

$$\boldsymbol{E}_R = \boldsymbol{\epsilon}_R E(x'' - \delta x) e^{i\boldsymbol{k}_R\cdot\boldsymbol{r} - i\omega t + i\phi(\theta_I,\theta_I^o)} \tag{10}$$

where $\boldsymbol{\epsilon}_R$ is a polarization vector, x'' is the coordinate perpendicular to the reflected wave vector $\boldsymbol{k}_R$, and the displacement $\delta x = -\frac{1}{k}\frac{d\phi}{d\theta_I}$ is determined by phase shift.

(e) Using the phase shift you computed, show that the lateral shift of the reflected in plane polarized beam is

$$D_\parallel = \frac{\lambda}{\pi}\frac{\sin\theta_I}{\sqrt{\sin^2\theta_I - \sin^2\theta_I^o}}\,\frac{\sin^2\theta_I^o}{\sin\theta_I^2 - \cos\theta_I^2\sin^2\theta_I^o} \tag{11}$$



Problem 4. Reflection of a Gaussian Wave Packet Off a Metal Surface:

In class we showed that the amplitude reflection coefficient from a good conductor ($\omega \ll \sigma$) for a plane wave of wavenumber $k = \omega/c$ is

$$\frac{H_R(k)}{H_I(k)} = 1 - \sqrt{\frac{2\mu\omega}{\sigma}}(1-i) \simeq \left(1 - \sqrt{\frac{2\mu\omega}{\sigma}}\right) e^{i\phi(\omega)}\,, \tag{12}$$

where the phase is for $\omega \ll \sigma$:

$$\phi(\omega) \simeq \sqrt{\frac{2\mu\omega}{\sigma}}\,. \tag{13}$$

Consider a Gaussian wave packet with average wave number k_o centered at $z = -L$ at time $t = -L/c$ which travels towards a metal plane located at $z = 0$ and reflects. Show that the time at which the center of the packet returns to $z = -L$ is given by

$$t = \frac{L}{c} + \frac{\mu\delta_o}{2c} \tag{14}$$

where the time delay is due to the phase shift $d\phi(\omega_o)/d\omega$, and $\delta_o = \sqrt{2c/\sigma\mu k_o}$ is the skin depth.

Problem 5. Jackson (7.16)

Plane waves propagate in a homogeneous, nonpermeable anisotropic dielectric. The dielectric of the crystal is characterized by a tensor ϵ_{ij}, *i.e.* $D_i = \epsilon_{ij} E_j$ and $j_i = \chi_{ij} \partial_t E_j$ with $\epsilon_{ij} = \delta_{ij} + \chi_{ij}$. But, if the coordinate axes are chosen as the principal axes of ϵ_{ij}, the components of displacement along these axes are related to the electric-field components by $D_i = \epsilon_i E_i$ ($i = 1, 2, 3$) where ϵ_i are the eigenvalues of the matrix ϵ_{ij}.

(a) Show that plane waves in the crystal with frequency ω and wave vector $\boldsymbol{k}$ must satisfy

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) + \frac{\omega^2}{c^2} \boldsymbol{D} = 0 \tag{15}$$

(b) (**Optional**) Show that for a given $\boldsymbol{k} = k\,\hat{\boldsymbol{k}}$, Eq. (15) may be regarded as a generalized eigenvalue problem[1] for the the eigen frequency ω_a and corresponding eigenvector $\boldsymbol{E}_a$. Show that their are two non-zero eigenmodes (i.e. why not three), and by appealing to general theorems of linear algebra (discussed in the footnote) determine the corresponding orthogonality relations between the eigenvectors.

(c) Show that for a given wave vector $\boldsymbol{k} = k\,\hat{\boldsymbol{k}}$ the phase velocities $v = \omega/k$ that satisfy the so call Fresnel equation

$$\sum_{i=1}^{3} \frac{(\hat{k}_i)^2}{v^2 - v_i^2} = 0 \tag{16}$$

where $v_i = c/\sqrt{\epsilon_i}$ is a velocity of a light wave propagating along a principal axis, and $(\hat{k}_i)$ is the components of $\hat{\boldsymbol{k}}$ along the ith principal axis.

(d) Show that $\boldsymbol{D}_a \cdot \boldsymbol{D}_b = 0$, where $\boldsymbol{D}_a$ and $\boldsymbol{D}_b$ are the electric displacements associated with the two modes of propagation. (Hint: use the eigenvalue equation satisfied by $\boldsymbol{E}$ and directly calculate $\boldsymbol{D}_a \cdot \boldsymbol{D}_b$.)

[1] The generalized eigenvalue problems is essentially the same as the ordinary eigenvalue problem. The (generalized) eigenvalue problem is $A\boldsymbol{v} = \lambda B \boldsymbol{v}$ where A and B are hermitian matrices, which can only be satisfied if $\det(A - \lambda B) = 0$. The matrix B is positive definite, $\boldsymbol{v}^\dagger B \boldsymbol{v} > 0$ for all v, and serves as a kind of measure for calculating the norms of vectors. The eigenvectors $\boldsymbol{v}_a$ are orthogonal with respect to a weight set by B, *i.e.* $\boldsymbol{v}_a^\dagger B \boldsymbol{v}_b = 0$ for $a \neq b$. It is often useful (as in this case) to work in an eigenbasis of B. The proofs of these statements are nearly identical to the proofs of the usual statements.