13 Radiation from Relativistic Charged Particles

13.1 Basic equations

(a) We wrote down the wave equations in the covariant gauge:

$$-\Box\Phi = \rho(t_o, \boldsymbol{r}_o) \tag{13.1}$$

$$-\Box\boldsymbol{A} = \boldsymbol{J}(t_o, \boldsymbol{r}_o)/c \tag{13.2}$$

(b) Then we used the green function of the wave equation

$$G(t, r|t_o r_o) = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|}\delta(t - t_o + \frac{|\boldsymbol{r} - \boldsymbol{r}_o|}{c}) \tag{13.3}$$

to determine the potentials $(\Phi, \boldsymbol{A})$ with the current

$$\frac{J^\mu}{c} = (\rho, \frac{\boldsymbol{J}}{c}) = (q\,\delta^3(\boldsymbol{r}_o - \boldsymbol{r}_*(t_o))\ ,\ q\,\frac{\boldsymbol{v}(t_o)}{c}\delta^3(\boldsymbol{r}_o - \boldsymbol{r}_*(t_o))) \tag{13.4}$$

This yields the Lienard-Wiechert potentials

$$\Phi = \frac{q}{4\pi|\boldsymbol{r} - \boldsymbol{r}_*(T)|}\frac{1}{1 - \boldsymbol{n}\cdot\boldsymbol{\beta}(T)} \Longrightarrow \frac{q}{4\pi r}\frac{1}{1 - \boldsymbol{n}\cdot\boldsymbol{\beta}(T)} \tag{13.5}$$

$$\boldsymbol{A} = \frac{q}{4\pi|\boldsymbol{r} - \boldsymbol{r}_*(T)|}\frac{\boldsymbol{\beta}(T)}{1 - \boldsymbol{n}\cdot\boldsymbol{\beta}(T)} \Longrightarrow \frac{q}{4\pi r}\frac{\boldsymbol{\beta}(T)}{1 - \boldsymbol{n}\cdot\boldsymbol{\beta}(T)} \tag{13.6}$$

where the retarded time is

$$T(t, r) = t - \frac{|\boldsymbol{r} - \boldsymbol{r}_*(T)|}{c} \Longrightarrow T(t, r) = t - \frac{r}{c} + \frac{\boldsymbol{n}\cdot\boldsymbol{r}_*(T)}{c} \tag{13.7}$$

The terms after the Longrightarrow indicate the far field limit

(c) The Lienard Wiechert potential can also be obtained by integrating over $\boldsymbol{r}_o$ in Eq. (11.8).

(d) The factor "collinear facor" (my name), or dT/dt

$$\frac{dT}{dt} = \frac{1}{(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})} \tag{13.8}$$

$$\frac{dT}{d\boldsymbol{r}^i} = \frac{1}{(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})}\frac{-\boldsymbol{n}_i}{c} \tag{13.9}$$

is quite important. We gave a physical interpretation of it in class. If a wave form is *observed* to have a time scale of Δt, then the *formation time* of the wave, ΔT, is

$$\Delta T = \frac{dT}{dt}\Delta t = \frac{\Delta t}{1 - \boldsymbol{n}\cdot\boldsymbol{\beta}} \tag{13.10}$$

In particular, a fourier component with frequency ω in the observed wave was formed over the time

$$\Delta T \sim \frac{1}{\omega(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})} \tag{13.11}$$

(e) In the ultrarelativistic limit $1/(1-\beta\cos\theta)$ is often approximated for $\theta \ll 1$ and for ultra-relativistic particles $1-\beta \simeq 1/2\gamma^2$

$$\frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}} = \frac{2\gamma^2}{1+(\gamma\theta)^2} \tag{13.12}$$

(f) The magnetic and electric fields can be determined from $\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A}_{\rm rad} - \nabla\Phi$. As discussed in a separate note ("retarded_time.pdf"), In the far field limit this is the same as computing

$$\boldsymbol{E}(t,r) = \boldsymbol{n}\times\boldsymbol{n}\times\frac{1}{c}\partial_t\boldsymbol{A}_{\rm rad}(T) \tag{13.13a}$$

$$= \boldsymbol{n}\times\boldsymbol{n}\times\frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}}\,\frac{1}{c}\frac{\partial}{\partial T}\boldsymbol{A}_{\rm rad}(T) \tag{13.13b}$$

$$= \frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}}\,\frac{1}{c}\frac{\partial}{\partial T}\left[\frac{q}{4\pi r}\,\frac{\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{\beta}}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}}\right]_{\rm ret} \tag{13.13c}$$

$$= \frac{q}{4\pi r c^2}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^3}\right]_{\rm ret} \tag{13.13d}$$

The $[]_{\rm ret}$ indicates that the velocity and acceleration are to be evaluated at the retarded time $T(t,r)$.

The magnetic field is

$$\boldsymbol{B} = \boldsymbol{n}\times\boldsymbol{E} \tag{13.14}$$

For below, it is worth noting below that

$$\frac{1}{c}\frac{\partial}{\partial T}\left[\boldsymbol{n}\times\boldsymbol{n}\times A_{\rm rad}\right] = \frac{1}{c}\frac{\partial}{\partial T}\left[\frac{q}{4\pi r}\,\frac{\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{\beta}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})}\right] \tag{13.15}$$

$$= \frac{q}{4\pi r c^2}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2}\right]_{\rm ret} \tag{13.16}$$

(g) We will often be interested in the frequency distribution of the radiation.

$$\boldsymbol{E}(\omega,r) \equiv \int_{-\infty}^{\infty} dt\, e^{i\omega t}\boldsymbol{E}(t,r) \tag{13.17a}$$

$$= \frac{q\, e^{i\omega r/c}}{4\pi r c^2}\int_{-\infty}^{\infty} dT\, e^{i\omega(T-\boldsymbol{n}\cdot\boldsymbol{r}_*(T)/c)}\,\frac{\boldsymbol{n}\times(\boldsymbol{n}-\boldsymbol{\beta})\times\boldsymbol{a}}{(1-\boldsymbol{n}\cdot\boldsymbol{\beta})^2} \tag{13.17b}$$

$$= \frac{q\,(-i\omega\, e^{i\omega r/c})}{4\pi r c}\int_{-\infty}^{\infty} dT\, e^{i\omega(T-\boldsymbol{n}\cdot\boldsymbol{r}_*(T)/c)}\;\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{\beta} \tag{13.17c}$$

We are computing the fourier transfrom of $\boldsymbol{E}_{\rm rad}(t,\boldsymbol{r})$ to find $\boldsymbol{E}_{\rm rad}(\omega,r)$. Changing variables to integrate over T instead of t yields Eq. (13.17b) with Eq. (13.13d). Integrating by parts using Eq. (13.15) yields Eq. (13.17c). This final form Eq. (13.17c) is often the most convenient, but sometimes it is just easier to use Eq. (13.17b) which shows explicity the dependence on acceleration.

Observables in the far field

(a) The energy per time per solid angle received at the detector is

$$\frac{dW}{dt d\Omega} = \frac{dP(t)}{d\Omega} = r^2\boldsymbol{S}\cdot n \tag{13.18}$$

$$= c|r\boldsymbol{E}|^2 \tag{13.19}$$

This is what you want to know if you want to find out if the detector will burn up.

(b) We often wan't to know how much energy was radiated over a given period of acceleration, $T_1 \ldots T_2$. For example how much energy was lost by the particle as it moved through one complete circle. Then we want to evaluate the energy radiated per retarded time from T_1 up to the time it completes the circle T_2

$$\frac{dW}{dT d\Omega} = \frac{dP(T)}{d\Omega} = r^2 \boldsymbol{S} \cdot \boldsymbol{n} \frac{dt}{dT} \tag{13.20}$$

$$= c|rE|^2 (1 - \boldsymbol{n} \cdot \boldsymbol{\beta}) \tag{13.21}$$

(c) We are also interested in the frequency distribution of the emitted radiation. The energy per $d\omega/(2\pi)$ per solid angle is

$$(2\pi) \frac{dW}{d\omega d\Omega} \equiv c|rE(\omega, r)|^2 \tag{13.22}$$

Since the sign of the ω is without significance (for real fields such as the electromagnetic fields), we sometimes use

$$\frac{dI}{d\omega d\Omega} \equiv \frac{c|rE(\omega, r)|^2}{2\pi} + \frac{c|rE(-\omega, r)|^2}{2\pi} = \frac{c|rE(\omega, r)|^2}{\pi} \tag{13.23}$$

So that

$$\frac{dW}{d\Omega} = \int_0^\infty \frac{dI}{d\omega d\Omega} \tag{13.24}$$

(d) The energy spectrum can be interperted as the average number of photons per frequency per solid angle

$$\frac{dI}{d\omega d\Omega} = \hbar \omega \frac{d\overline{N}}{d\omega d\Omega} \tag{13.25}$$

13.2 Relativistic Larmour

(a) For a particle undergoing arbitrary relativistic motion, we evaluated the energy per retarded time per solid angle

$$\frac{dP(T)}{d\Omega} = \frac{q^2}{16\pi^2 c^3} \frac{|\boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{\beta}) \times \boldsymbol{a}|^2}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^5} \tag{13.26}$$

(b) Integrating over angles we get

$$P(T) = \frac{dW}{dT} = \frac{q^2}{4\pi} \frac{2}{3c^3} \gamma^6 \left[a_\parallel^2 + \frac{a_\perp^2}{\gamma^2} \right] \tag{13.27}$$

where $a_\parallel$ is the projection of $\boldsymbol{a} = d^2\boldsymbol{x}/dt^2$ along the direction of motion, and $a_\perp$ is the component of $\boldsymbol{a}$ perpendicular to the direction of motion, *i.e.* for $\boldsymbol{v}$ in the z direction

$$\boldsymbol{a} = (a_\perp^x, a_\perp^y, a_\parallel) \tag{13.28}$$

(c) The acceleration four vector is

$$\mathscr{A}^\mu = \frac{d^2 x^\mu}{d\tau^2} \tag{13.29}$$

For a paraticle moving along in the z-direction, the acceleration in the particle's locally inertial frame (i.e. the frame that is instantaneously moving with the particle) is

$$(\mathscr{A}^0, \mathscr{A}^1, \mathscr{A}^2, \mathscr{A}^3)\big|_{\text{rest frame}} = (0, \alpha_\perp^x, \alpha_\perp^y, \alpha_\parallel) \tag{13.30}$$

While in the lab frame $\mathscr{A}^\mu$ is found by boosting this result. The acceleration $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$ is found from this result and the definition of propper time $d\tau = dt/\gamma$,

$$\boldsymbol{a} = (a_\perp^x, a_\perp^y, a_\parallel) = (\gamma^2 \alpha_\perp^x,\ \gamma^2 \alpha_\perp^y,\ \gamma^3 \alpha_\parallel) \tag{13.31}$$

You should be able to prove this. The relativistic Larmour fourmula can then be written

$$P(T) = \frac{q^2}{4\pi} \frac{2}{3c^3} \mathscr{A}_\mu \mathscr{A}^\mu \tag{13.32}$$

(d) For straight line acceleration at very large γ, we found that that the radiation is emitted within a cone of order

$$\Delta\Theta \sim 1/\gamma\,. \tag{13.33}$$

For θ very small $\theta \sim 1/\gamma$ we found,

$$\frac{dP(T)}{d\Omega} = \frac{2q^2}{\pi^2}\frac{a^2}{c^3}\gamma^8 \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}\,. \tag{13.34}$$

You should feel comfortable deriving this result.