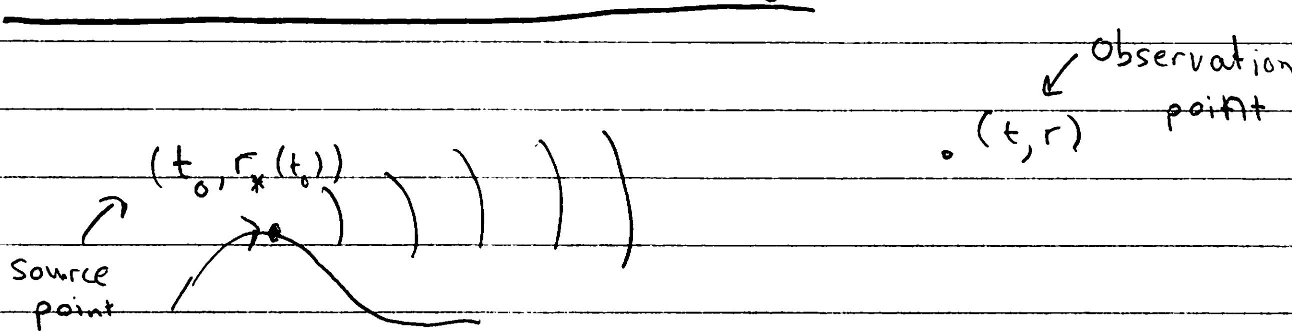
Radiation From Relativistic Charges



$$-\Box \varphi = \rho$$

$$-\Box \vec{A} = \vec{j}/c$$

Using the Grn fcn

$$G(t r | t_0 r_0) = \frac{\Theta(t - t_0)}{4\pi |\vec{r} - \vec{r}_0|} \delta(t - t_0 - |\vec{r} - \vec{r}_0|/c)$$

So

$$\varphi(t, \vec{r}) = \int dt_0 \, d^3 r_0 \; \delta\left(t - t_0 - \frac{|\vec{r} - \vec{r}_0|}{c}\right) \frac{\rho(t_0, r_0)}{4\pi |\vec{r} - \vec{r}_0|}$$

$$\vec{A}(t, \vec{r}) = \int dt_0 \, d^3 r_0 \; \delta\left(t - t_0 - \frac{|\vec{r} - \vec{r}_0|}{c}\right) \frac{\vec{j}/c \, (t_0, r_0)}{4\pi |\vec{r} - \vec{r}_0|}$$

For point charge

$$\rho(t_0, r_0) = e \, \delta^3(\vec{r}_0 - \vec{r}_*(t_0))$$

$$\vec{J}(t_0, r_0) = e V(t_0) \, \delta^3(\vec{r}_0 - r_*(t_0))$$

Doing the $d^3 r_0$ integral

$$\varphi(t,r) = \int dt_0\, \delta\left(t - t_0 - \frac{|\vec{r} - \vec{r}_*(t_0)|}{c}\right) \frac{e}{4\pi R} \qquad R \equiv |\vec{r} - r_*(t_0)|$$

$$\vec{A}(t,\vec{r}) = \int dt_0\, \delta\left(t - t_0 - \frac{|\vec{r} - \vec{r}_*(t_0)|}{c}\right) \frac{e\vec{v}(t_0)}{4\pi R}$$

Now we do the time integral, for each value of $\vec{t}, \vec{r}$ only one time moment $t_0 = T$ will contribute

$$T = t - \frac{|\vec{r} - r_*(T)|}{c} = \text{retarded time or source time}$$

Using $\delta(f(t_0)) = \dfrac{\delta(t_0 - T)}{|f'(T)|}$ with $f(t_0) = t - t_0 - \dfrac{|r - r_*(t_0)|}{c}$

we have

$$\frac{df}{dt_0} = -1 - \frac{1}{c}\frac{d}{dt_0}\left((\vec{r} - r_*(t_0))^2\right)^{1/2} = -1 + \vec{n} \cdot \frac{V}{c}(t_0)$$

$$\text{with} \quad \vec{n} \equiv \frac{\vec{r} - \vec{r}_*(t_0)}{|\vec{r} - r_*(t_0)|}$$

And so

$$\frac{1}{|f'(T)|} = \frac{1}{\left(1 - \frac{n}{c}\cdot V(T)\right)}$$

So we are led to

$$\boxed{\begin{aligned} \varphi(t,\vec{r}) &= \frac{e}{4\pi R}\,\frac{1}{\left(1-\frac{n\cdot v(T)}{c}\right)} \\ \vec{A}(t,\vec{r}) &= \frac{e\,\vec{v}(T)/c}{4\pi R}\,\frac{1}{\left(1-\vec{n}\cdot\frac{\vec{v}(T)}{c}\right)} \end{aligned}}$$

These are known as the Lienard - Wiechert Potentials

Let us specialize to the far field

- $\frac{1}{R} = \frac{1}{|\vec{r}-r_*(T)}\simeq \frac{1}{r}$
- $T = t - \frac{|\vec{r}-\vec{r}_*(T)|}{c} \simeq \boxed{t - \frac{\vec{n}}{c}\cdot(\vec{r}-r_*(T)) = T}$
- $\vec{n}\simeq\frac{\vec{r}}{|\vec{r}|}$

retarded time in far field
this is an implicit function
of t, r

Problem

- Show that $\frac{\partial T}{\partial t} = \frac{1}{(1-n\cdot V/c)}$ and $\frac{\partial T}{\partial r^i} = \frac{1}{(1-\frac{n}{c}\cdot V)}(-n_i/c)$

Interpret $\frac{\Delta T}{\Delta t}$ physically by drawing a picture.

Solution

- use implicit differentiation

① $T = t - \frac{\vec{n}}{c}\cdot(\vec{r} - r_*(T))$

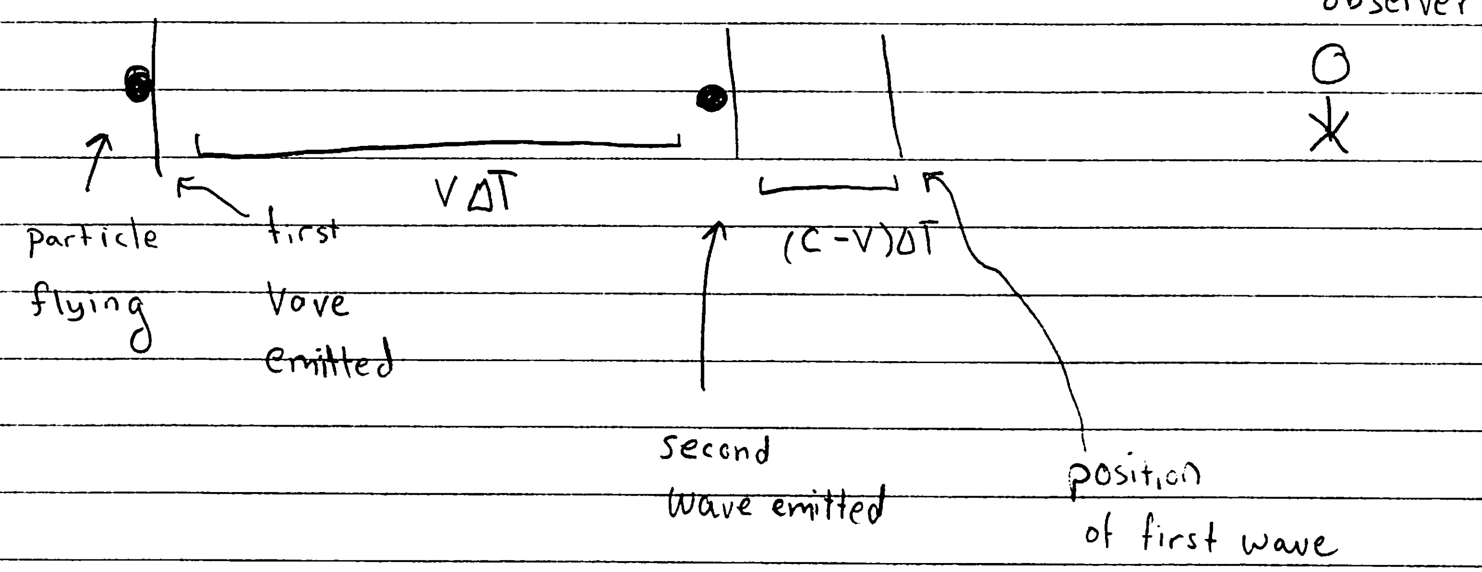
$$\frac{\partial T}{\partial t} = 1 + \frac{n}{c}\cdot\underbrace{\frac{\partial r_*}{\partial T}}_{\text{velocity}}\frac{\partial T}{\partial t} \Longrightarrow \frac{\partial T}{\partial t} = \frac{1}{1-\frac{n}{c}\cdot V_*(T)}$$

② $$\frac{\partial T}{\partial r^i} = -\frac{n_i}{c} + \frac{n}{c}\cdot\frac{\partial r_*}{\partial T}\frac{\partial T}{\partial r^i} \Rightarrow \frac{\partial T}{\partial r^i} = \frac{1}{(1-\frac{n}{c}\cdot V_*(T))}\,-\vec{n}_i/c$$

So then we have an interpretation of $\frac{1}{(1-\vec{n}\cdot\vec{v}/c)}$.

First note that this factor can be very large, if the observation direction is parallel to $\vec{v}$ and $\vec{v} \simeq c$.

Physical interpretation of $\partial T/\partial t = 1/(1-n\cdot\vec{v}/c)$



So the observer measures the time difference between the signals to be:

$$\Delta t = \frac{(c-v)\,\Delta T}{c}$$

$$\Delta t = \left(1-\frac{v}{c}\right)\Delta T$$

$$\frac{1}{\left(1-\frac{v}{c}\right)}\,\Delta t = \Delta T$$

So

$$\frac{\Delta T}{\Delta t} = \frac{\text{formation time of radiation}}{\text{observation time of radiation}} = \frac{1}{\left(1-\frac{\vec{n}\cdot\vec{v}}{c}\right)}$$

Fields of Lienard-Wiechert

Now we can compute the Electric Field

$$\vec{E}_{rad} = \vec{n} \times \vec{n} \times \frac{1}{c}\frac{\partial \vec{A}_{rad}}{\partial t}$$

So first we relate $\partial A/\partial t$ and $\partial A/\partial T$:

$$\frac{1}{c}\frac{\partial \vec{A}_{rad}}{\partial t} = \frac{1}{c}\frac{\partial A_{rad}}{\partial T}\frac{\partial T}{\partial t} = \frac{1}{c}\frac{\partial A_{rad}}{\partial T}\frac{1}{(1-n\cdot\frac{\vec{v}}{c})}$$

Using:

$$\vec{A}_{rad} = \frac{e}{4\pi r}\frac{\vec{v}(T)/c}{(1-\vec{n}\cdot\vec{V}/c)}$$

$$\frac{1}{c}\frac{\partial A_{rad}}{\partial T} = \frac{e}{4\pi r c^2}\left[\frac{\vec{a}(T)}{(1-n\cdot V/c)} + \frac{\vec{\beta}\,(n\cdot\vec{a})}{(1-n\cdot V/c)^2}\right]$$

Use $a\times(b\times c) = b(ac) - (ab)c$

$$= \frac{e}{4\pi r c^2}\frac{1}{(1-n\cdot V/c)^2}\left[\vec{a} + \vec{n}\times\vec{\beta}\times\vec{a}\right]$$

Then $\vec{n}\times\vec{n}\times\frac{1}{c}\frac{\partial \vec{A}_{rad}}{\partial t}$ is (use $n\times n\times(\underbrace{n\times\beta\times\vec{a}}_{\text{aleady transverse}}) = -n\times\beta\times a$)

$$\boxed{\vec{E}_{rad} = \frac{e}{4\pi r c^2}\frac{1}{(1-\vec{n}\cdot\vec{V}/c)^3}\left[\vec{n}\times(\vec{n}-\vec{\beta})\times\vec{a}\right]}$$

$$\boxed{\vec{B}_{rad} = \vec{n} \times \vec{E}_{rad}}$$

It is understood that $v(T)$ and $a(T)$ are to be evaluated at the retarded time

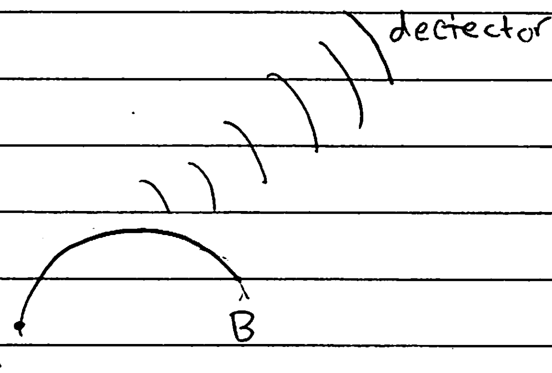
$$T = t - \frac{\vec{n}}{c} \cdot (\vec{r} - r_*(T))$$

The Power

Radiated Power

$$\frac{dP(t)}{d\Omega} = \frac{dW}{dt\,d\Omega} = r^2 \vec{S}\cdot\vec{n}$$

this is what you want to know, if you want to know if the detector will burn up.



- One often wants to know how much energy was radiated away as the particle moved from A (labelled by $(T_A, r_*(T_A))$ and B (labelled by $(T_B, r_*(T_B))$). Then you want to know

$$\frac{dP(T)}{d\Omega} = \frac{dW}{dT\,d\Omega} = \frac{dW}{dt\,d\Omega}\frac{dt}{dT}$$

Using

$$\frac{dT}{dt} = \frac{1}{(1-n\cdot\beta(T))} \qquad S = c\,E^2\,\vec{n}$$

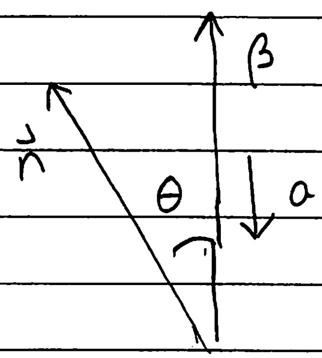
We have

$$\frac{dP(T)}{d\Omega} = |r\,E|^2\,(1-n\cdot v(T)/c)$$

$$= \frac{e^2}{16\pi^2 c^3}\frac{|n\times(n-\beta)\times\vec{a}|^2}{(1-n\cdot\beta(T))^5}$$

Radiated Power $\vec{a}$ parallel to $\vec{\beta}$

Then lets take the simplest case. A particle moving relativistically but decelerating along the motion:



Then, $\vec{n}\cdot\vec{\beta} = \beta\cos\theta$

$$|\vec{n}\times\vec{n}\times\vec{a}| = a_T = a\sin\theta$$

So

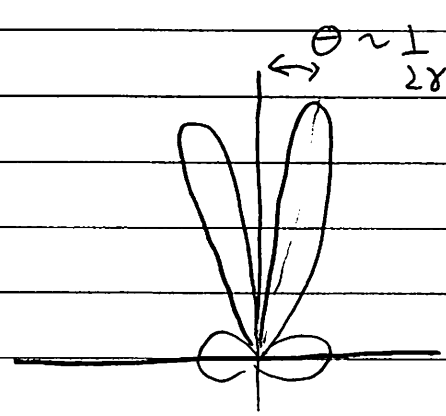
$$\boxed{\frac{dP(T)}{d\Omega} = \frac{e^2 a^2}{16\pi^2 c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}}$$

Comments:

① For non-relativistic motion, get the Larmour result
$\beta \ll 1$

$$\frac{dP(T)}{d\Omega} = \frac{e^2}{16\pi^2 c^3} a^2 \sin^2\theta$$

② For $\beta \to 1$ and $\theta \to 0$, $\frac{1}{(1-\beta\cos\theta)}$ gets large, and the radiation is peaked in the direction of motion. For $\gamma \simeq 2$ this is plot



← Polar Plot of $\frac{dP}{d\Omega}$

Radiated Power $a \parallel$ to β pg. 2

Take the limit $\beta \to 1$, θ small, $\gamma^2 = \frac{1}{(1-\beta^2)} = \frac{1}{2(1-\beta)}$

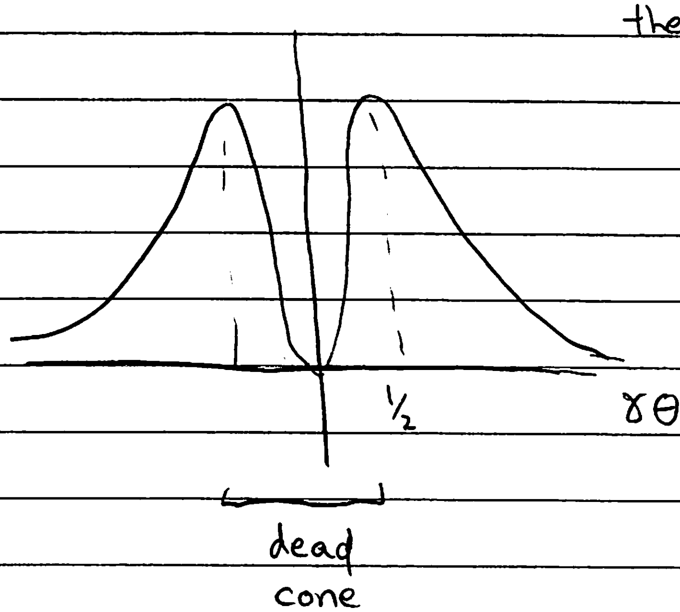
$$\frac{1}{(1-\beta\cos\theta)} \simeq \frac{1}{(1-\beta) + \frac{\theta^2}{2}}$$

$$\simeq \frac{1}{\frac{1}{2\gamma^2} + \frac{\theta^2}{2}} \simeq \frac{2\gamma^2}{(1+(\gamma\theta)^2)}$$

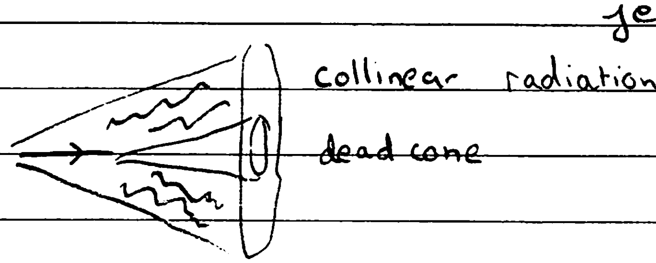
So then with $\sin\theta \simeq \theta$

$$\frac{dP}{d\Omega} = \frac{2e^2}{\pi^2}\frac{a^2}{c^3}\gamma^8 \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}$$

So the picture is take γ large ~ 100. Then the radiation is peaked in the forward direction $\theta \sim \frac{1}{100}$.



But, only transverse currents radiate. So in the direction of motion of the particle, there is no radiation. This is known as the dead-cone, and is characteristic of heavy quark jets.



Total Radiated Power $a \parallel$ to β

We can also compute the total power:

$$d\Omega = 2\pi \sin\theta \, d\theta \simeq 2\pi\,\theta\, d\theta \qquad (\theta \ll 1)$$

So

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2e^2}{\pi^2}\frac{a^2}{c^3}\gamma^6 \int_0^{\pi} \gamma\, d\theta\, \gamma\theta \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}$$

Let $x = \gamma\theta$ $\quad 2\pi$

$$P = \frac{4e^2}{\pi}\frac{a^2}{c^3}\gamma^6 \int_0^{\gamma\pi} x\, dx \frac{x^2}{(1+x^2)^5}$$

Now you can extend the upper limit $\gamma\pi \to \infty$ (γ large) and find

$$\boxed{P = \frac{e^2}{4\pi}\frac{2}{3}\frac{a_{\parallel}^2}{c^3}\gamma^6}$$

We will see that this is a special case of a relativistic generalization of the Larmour formula. Note that I put $a_{\parallel}$ because I have assumed that the acceleration is parallel to the velocity. In general,

Total Power pg. 1

the acceleration has a component parallel to the velocity $a_{\parallel}$, + perpendicular to the velocity $a_{\perp}$. The full generalization of Larmour is (see below)

$$(1) \qquad \boxed{P(T) = \frac{e^2}{4\pi}\frac{2}{3}\frac{\gamma^6}{c^3}\left[a_{\parallel}^2 + \frac{a_{\perp}^2}{\gamma^2}\right]}$$

Lienard-Wiechert 1898, predating relativity by seven years!

Proof of Lienard Wiechert (Brute force)

(Skip if pressed for time!)

$$\star \; P(T) = \int d\Omega \, \frac{e^2}{16\pi^2 c^3} \frac{|n \times (\vec{n} - \vec{\beta}) \times \vec{a}|^2}{(1 - n \cdot \beta)^5}$$

The simplest way is to take β along z-axis, and $\vec{a}$ in the x-z plane and then do all integrals

$$\vec{\beta} = (0, 0, \beta)$$

$$\vec{a} = (a_{\perp}, 0, a_{\parallel})$$

$$\cancel{\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)}$$

Analysis of Lienard-Wiechert Result

$$A^\mu = \frac{d^2 x^\mu}{d\tau^2}$$ = propper acceleration analyzed in homework

In LRF of particle (LRF = local rest frame

$$A^\mu = \begin{pmatrix} 0 \\ \alpha_{\parallel} \\ \alpha_{\perp} \end{pmatrix} \qquad A^\mu A_\mu = \alpha_{\parallel}^2 + a_{\perp}^2$$

Then hmwrk was to show

$$\gamma^3 a_{\parallel} = \alpha_{\parallel} \quad \text{and} \quad \gamma^2 a_{\perp} = \alpha_{\perp}$$

Then

$$A^\mu A_\mu = \gamma^6 \left[a_{\parallel}^2 + \frac{a_{\perp}^2}{\gamma^2} \right]$$ ← See prf at end of lecture

So

$$\frac{dW}{dT} = \frac{e^2}{4\pi} \frac{2}{3} A^\mu A_\mu$$

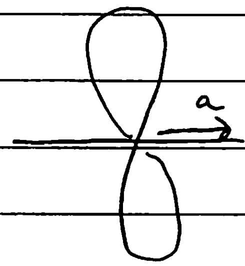
Total Power (Pure Thinking)

In retrospect could "guess" this result

Look at the emission in rest frame of particle

$$= A^\mu A_\mu \text{ in rest frame}$$

energy emitted → $\Delta E = \frac{e^2}{4\pi} \frac{2}{3c^2} a^2 \Delta t$



momentum emitted → $\Delta \vec{P} = 0$ ← Since radiation is emitted symmetricall and transverse to beam

$$\Delta t = \Delta t$$

$$\Delta x = 0$$

Then under boost

$$\underline{\Delta E} = \gamma \, \Delta E$$

$$\begin{pmatrix} \underline{\Delta t} \\ \underline{\Delta p} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \Delta E \\ \Delta p \end{pmatrix}$$

$$\underline{\Delta t} = \gamma \, \Delta t$$

And

$$\text{total power} = \frac{\underline{\Delta E}}{\underline{\Delta t}} = \frac{\Delta E}{\Delta t} = \text{invariant}$$

$$= \frac{e^2}{4\pi} \frac{2}{3c^3} \underbrace{A^\mu A_\mu}_{\text{true in all}}$$

Linear vs. Circular Accelerators

In general, since $p^\mu = m u^\mu$, and $A^\mu = du^\mu/d\tau$

$$\frac{dW}{d\tau} = \frac{e^2}{4\pi}\frac{2}{3} A^\mu A_\mu = \frac{e^2}{4\pi}\frac{2}{3}\frac{1}{m^2c^3}\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}$$

- Then for a linear accelerator where $d\vec{p}/dt$ is parallel to v

$$\frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt} \qquad \frac{dp^0}{d\tau} = \frac{d\sqrt{p^2+m^2}/c}{d\tau} = \frac{v}{c}\frac{dp}{d\tau} = \frac{\gamma v}{c}\frac{d\vec{p}}{dt}$$

So

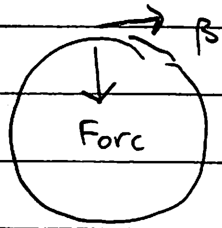
$$\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = -\left(\frac{dp^0}{d\tau}\right)^2 + \left(\frac{dp}{d\tau}\right)^2 = \left(\frac{d\vec{p}}{dt}\right)^2$$

So that the radiated energy grows with the applied force squared

$$\frac{dW}{d\tau} = \frac{e^2}{4\pi}\frac{2}{3m^2c^3}\left(\frac{d\vec{p}}{dt}\right)^2$$

and is independent of γ

- By contrast for a circular accelerator where $\frac{d\vec{p}}{dt}$ is perpendicular to $\vec{v}$. We have that:



$$\frac{dp^{\mu}}{d\tau}\frac{dp_{\mu}}{d\tau} = \frac{d\vec{p}_{\perp}}{d\tau}\cdot\frac{d\vec{p}_{\perp}}{d\tau} = \gamma^2\left(\frac{d\vec{p}_{\perp}}{dt}\right)^2$$

We have that

$$\boxed{\frac{dW}{dT} = \frac{e^2}{4\pi}\frac{2}{3\,m^2c^3}\,\gamma^2\left(\frac{d\vec{p}_{\perp}}{dt}\right)^2}$$

So the radiated power grows as γ^2!!/ This is becoming prohibitive at colliders today, and is a big reason for research into Linear accelerators