Lecture Goals

(1) Show that the maxwell equations in media lead to wave-eqns.

(2) Show that the solutions to the wave eqn are

$$\vec{E} = \vec{\mathcal{E}}\, e^{i\vec{k}\vec{x} - i\omega t}$$

$$\vec{H} = \vec{\mathcal{H}}\, e^{i\vec{k}\vec{x} - i\omega t}$$

where:

a) ω and $\vec{k}$ are related by the phase velocity

$$V_\phi = \frac{\omega}{k} = \frac{c}{n}$$

with $n = \sqrt{\mu\varepsilon}$ is the index of refraction

Note in mks units / HL conversion

$$\mu_{HL} = \frac{\mu_{mks}}{\mu_0} \qquad \varepsilon_{HL} = \frac{\varepsilon_{mks}}{\varepsilon_0}$$

b) $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ are orthogonal to $\vec{k}$ and each other

$$\vec{k} \cdot \vec{\mathcal{E}} = \vec{k} \cdot \vec{\mathcal{H}} = 0$$

Similarly

$$\mathcal{H} = \frac{1}{Z_r} \vec{k} \times \vec{\mathcal{E}}$$

Here $Z_r = \sqrt{\frac{\mu}{\varepsilon}}$ is the relative impedance

(In the MKS system, $Z_r = \sqrt{\mu/\varepsilon} \Big/ \sqrt{\mu_0/\varepsilon_0}$, and $\sqrt{\mu_0/\varepsilon_0} = 376\ \Omega$ is the impedance of the vacuum)

③ Understand that polarized light

$$\vec{E} = E_0 \vec{\varepsilon}_+ e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

with $\vec{\varepsilon}_+ = \frac{\vec{\varepsilon}_1 + i\vec{\varepsilon}_2}{\sqrt{2}}$ represent a beam of light with positive helicity

Plane Waves in Linear Matter

① $\nabla \cdot D = 0$

② $\nabla \times H = \frac{1}{c} \partial_t D$

③ $\nabla \cdot B = 0$

④ $-\nabla \times E = \frac{1}{c} \partial_t B$

Note a symmetry in absence of currents

$H \to -E$ and $D \to B$

In vacuum, $B \to -E$ and $E \to B$ (Electric - magnetic duality)

Then to derive the wave-eqn in linear matter we take curl of ② and use "bac - abc"

$$\nabla \times (\nabla \times H) = \frac{1}{c} \partial_t (\nabla \times \vec{D})$$

$$\nabla (\cancel{\nabla \cdot} \vec{H}) - \nabla^2 \vec{H} = -\frac{\mu \varepsilon}{c^2} \partial_t^2 H$$

$D = \varepsilon E$

$B = \mu H$

$$\boxed{\left(\frac{\mu \varepsilon}{c^2} \partial_t^2 - \nabla^2 \right) H = 0}$$

By symmetry

$$\boxed{\left(\frac{\mu \varepsilon}{c^2} \partial_t^2 - \nabla^2 \right) E = 0}$$

Now we pass from the time dependent wave-eqn to the time-indepent wave eqn

$$H(x,t) = e^{-i\omega t} H(x) \qquad E(x,t) = e^{-i\omega t} E(x)$$

To find the Helmholtz equations:

$$\boxed{\begin{aligned} \left[\omega^2 \left(\frac{\mu\varepsilon}{c^2}\right) + \nabla^2\right] \vec{H}(x) = 0 \\ \left[\omega^2 \left(\frac{\mu\varepsilon}{c^2}\right) + \nabla^2\right] \vec{E}(x) = 0 \end{aligned}}$$

This is an equation for the allowed $\underset{\text{eigen}}{}$frequencies and the corresponding modes. The general solution is a super-position of these modes.

Try

$$E = \vec{\mathcal{E}}\, e^{i\vec{k}\vec{x} - i\omega t}$$

$$\vec{H} = \vec{\mathcal{H}}\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

Here $\vec{\mathcal{E}}$ + $\vec{\mathcal{H}}$ are constant vectors. We will show they are $\perp$ to $\vec{k}$ and each other.

Find

$$-k^2 + \omega^2 \left(\frac{\mu\varepsilon}{c^2}\right) = 0 \quad \Rightarrow \quad \omega = \frac{ck}{n}$$

where

$n = \sqrt{\mu\varepsilon}$ is the index of refraction

Propeties of Plane Waves

The phase velocity of the wave is

$$V_\phi = \frac{\omega}{k} = \frac{c}{n}$$

Now the divergence Eqs

$$\nabla \cdot E = 0$$

$$\nabla \cdot B = 0$$

give rise to

$$\left.\begin{aligned} \vec{k} \cdot \vec{\mathcal{E}} &= 0 \\ k \cdot \mathcal{H} &= 0 \end{aligned}\right\}$$

Thus the vectors $\vec{\mathcal{E}}$ & $\vec{\mathcal{H}}$ are transvere to the beam

Finally we have:

$$\nabla \times E = -\frac{1}{c} \partial_t B$$

$$i\vec{k} \times \vec{\mathcal{E}} = \frac{i\omega}{c} \mu \vec{\mathcal{H}}$$

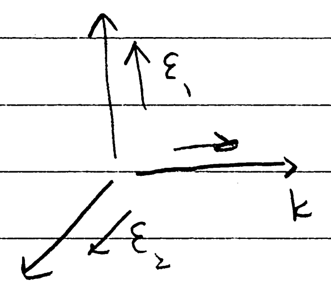
For $k = \omega/(c/n)$ find using $\hat{k} = \vec{k}/k$

$$\frac{1}{Z} \vec{k} \times \vec{\mathcal{E}} = \vec{\mathcal{H}} \qquad \text{with} \quad Z \equiv \sqrt{\frac{\mu}{\varepsilon}}$$

$\vec{\mathcal{H}}$ is $\frac{1}{Z}$ relative to $\vec{\mathcal{E}}$, or $\mathcal{B}$ is $\sqrt{\mu\varepsilon}$ relative $\vec{\mathcal{E}}$.

Summary of Polarization

- Construct two vectors which are orthogonal to $\vec{k}$



$$\vec{\varepsilon}_1 \cdot \vec{\varepsilon}_2 = 0 \qquad \varepsilon_1^2 = \varepsilon_2^2 = 1$$

$$\vec{\varepsilon}_1 \cdot \vec{k} = \vec{\varepsilon}_2 \cdot \vec{k} = 0$$

Found

$$\vec{\mathcal{E}} = \vec{\varepsilon}_1 E_0 \qquad \vec{\mathcal{H}} = \frac{1}{Z} \hat{k} \times \vec{\mathcal{E}}_0 = \frac{1}{Z} \vec{\varepsilon}_2 E_0$$

or the reverse: $\mathcal{E} \to \vec{H} \qquad \mathcal{H} \to -\mathcal{E}$

$$\vec{\mathcal{E}} = -\vec{\varepsilon}_2 E_0 \qquad \mathcal{H} = \frac{1}{Z} (+\vec{\varepsilon}_1) E_0$$

In general we take a complex superposition

$$\vec{E} = (\vec{\varepsilon}_1 E_1 + \vec{\varepsilon}_2 E_2) e^{i\vec{k}\cdot\vec{x} - i\omega t} \qquad E_1, E_2 \text{ complex}$$

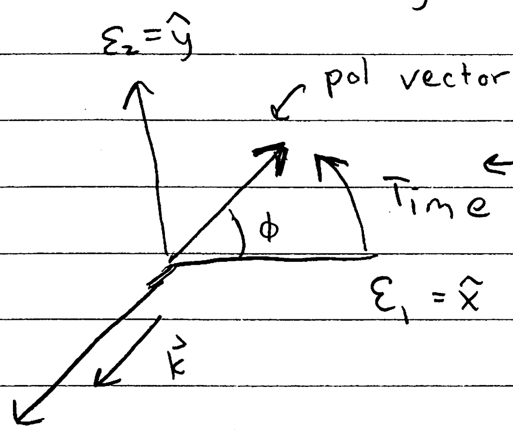
① If E_1 and E_2 are in phase, then the light is linearly polarized.

② If E_1 and E_2 are out phase by $90°$ (and equal in magnitude) the light is circularly polarized

$$\vec{E} = E_0 \frac{(\vec{\varepsilon}_1 \pm i \vec{\varepsilon}_2)}{\sqrt{2}} e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

So

$$\mathrm{Re}\,\vec{E} = \frac{1}{\sqrt{2}}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_0 \cos(kz - \omega t) \\ \mp E_0 \sin(kz-\omega t) \end{pmatrix}\frac{1}{\sqrt{2}}$$



← So from the picture we have,

$\frac{\mathcal{E}_1 + i\mathcal{E}_2}{\sqrt{2}}$ has positive helicity

while

$\frac{\mathcal{E}_1 - i\mathcal{E}_2}{\sqrt{2}}$ has negative helicity

To see this, take $z = 0$:

$$\phi(t) = \tan^{-1} E_y / E_x = \tan^{-1}\left(\mp \frac{\sin(-\omega t)}{\cos(-\omega t)}\right) = \pm \omega t$$

Now we define:

$$\mathcal{E}_+ \equiv \frac{\mathcal{E}_1 + i\mathcal{E}_2}{\sqrt{2}} \qquad \mathcal{E}_- \equiv \frac{\mathcal{E}_1 - i\mathcal{E}_2}{\sqrt{2}}$$

So $\vec{\mathcal{E}}_+ \cdot \vec{\mathcal{E}}_-^* = 0$, $\vec{\mathcal{E}}_+ \cdot \vec{\mathcal{E}}_+^* = 1$, and $\vec{\mathcal{E}}_- \cdot \vec{\mathcal{E}}_-^* = 1$.

$\vec{E} = E_0\, \vec{\mathcal{E}}_+\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$ has positive helicity while

$$\vec{E} = E_0 \vec{\varepsilon}_- e^{i\vec{k}\cdot\vec{x} - i\omega t}$$ has negative helicity

In general a plane wave

$$\vec{E} = (E_+ \vec{\varepsilon}_+ + E_- \vec{\varepsilon}_-) e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

is a superposition of positive and negative helicities.

Time Averaging

$$\langle \vec{S} \rangle = c \left\langle \left(\mathrm{Re}\, \mathcal{E} e^{-i\omega t}\right) \times \left(\mathrm{Re}\, \mathcal{H} e^{-i\omega t}\right) \right\rangle$$

$$= c \left\langle \frac{\left(\mathcal{E} e^{-i\omega t} + \mathcal{E}^* e^{i\omega t}\right)}{2} \times \frac{\left(\mathcal{H} e^{-i\omega t} + \mathcal{H}^* e^{i\omega t}\right)}{2} \right\rangle$$

$$= \frac{c}{4}\left(\mathcal{E} \times H^* + \mathcal{E}^* \times \mathcal{H}\right) + \cancelto{0}{\left\langle \text{oscillating terms} \propto e^{-2i\omega t} \right\rangle}$$

$$= \frac{c}{2}\,\mathrm{Re}\left(\mathcal{E} \times \mathcal{H}^*\right)$$

In general take half the Real part, with complex conjugate on second term.

Similarly

$$u_{em} = \frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2$$

So

$$\langle u_{em} \rangle = \frac{1}{2}\,\mathrm{Re}\Big(\overbrace{\frac{\varepsilon}{2}\vec{\mathcal{E}} \cdot \mathcal{E}^*}^{\text{electric}} + \overbrace{\frac{\mu}{2}\mathcal{H} \cdot \mathcal{H}^*}^{\text{magnetic}} \Big)$$

Electric and magnetic energies are equal

$$= \frac{1}{2}\varepsilon\, \vec{\mathcal{E}} \cdot \vec{\mathcal{E}}^*$$

we used $\vec{\mathcal{H}} = \frac{1}{Z}\hat{k} \times \vec{\mathcal{E}}$ and dropped the Re part because $\vec{\mathcal{E}} \cdot \vec{\mathcal{E}}^*$ is real

Note also since $\mathcal{H} = \frac{1}{Z} \hat{k} \times \mathcal{E}$

$$\langle \vec{S} \rangle = \frac{c}{2Z} |\mathcal{E}|^2 \hat{k} = \frac{c}{n} \langle u \rangle \hat{k}$$

This makes sense the Poynting vector is just the energy density times the speed, c/n:

$$\langle \vec{S} \rangle = \frac{\text{energy}}{\text{area time}} = \frac{\text{energy}}{\text{vol}} \cdot \frac{\text{distance}}{\text{time}}$$

Note also

$$\langle T^{ij}_E \rangle = \frac{1}{2} \mathrm{Re} \left(-E^i E^{j*} + \frac{1}{2} E \cdot E^* \delta^{ij} \right)$$

$$= \frac{1}{2} \left(\frac{-E^i E^{j*} - E^{i*} E^j}{2} + \frac{1}{2} E \cdot E^* \delta^{ij} \right)$$