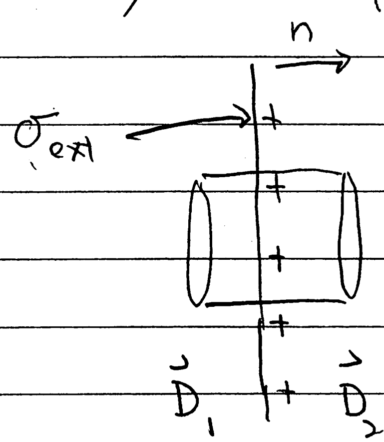
Boundary Conditions

Now we want to solve

$$\nabla \cdot D = \rho_{ext}$$
$$\nabla \times E = 0$$

Before we can do so we need boundary conditions

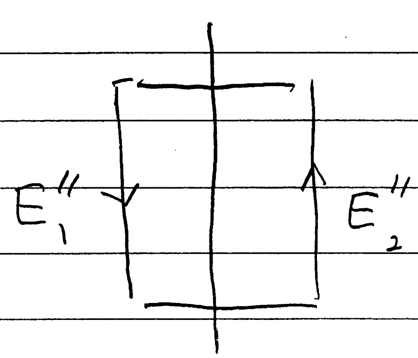
From, $\nabla \cdot D = \rho_{ext}$, we apply Gaus law to this and find



$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma_{ext}$$

i.e the normal components of D jump across the surface

From $\nabla \times E = 0$ or $\oint \vec{E} \cdot d\ell = (E_2^{\parallel} - E_1^{\parallel})\,\ell = 0$

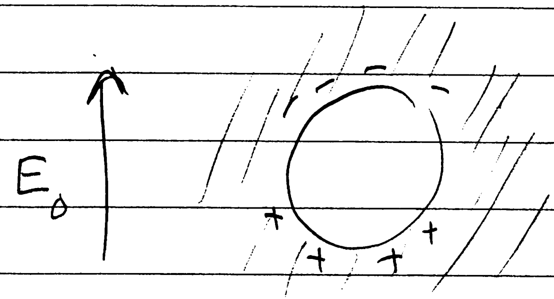


we have the continuity of parallel electric field component

$$E_2^{\parallel} = E_1^{\parallel}$$

A model problem. A spherical cavity (Jackson 4.4)

Consider a cavity in a dielectric of radius a in an external field



The polarized charge we will find is sketched. (Why does it look like this?)

Solution

$\nabla \cdot D = \rho$ and for constant ε $\nabla \cdot (\varepsilon \nabla \varphi) = \rho$

$\nabla \times E = 0 \implies E = -\nabla \varphi$

So outside the sphere $\rho = 0$ and $-\nabla^2 \varphi = 0$

$$\varphi^{out} = \sum_{\ell} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

while inside

$$\varphi^{in} = \sum_{\ell} \left(C_\ell r^\ell + \cancelto{0}{\frac{D_\ell}{r^\ell}} \right) P_\ell(\cos\theta)$$

The only non-vanishing A_ℓ, is A_1.

$$\varphi^{out} = -E_0 r\cos\theta + \sum_{\ell} \frac{B_\ell}{r^{\ell+1}} \cos\theta$$

Now on surface $r=a$ we have:

$$\vec{n}\cdot D^{out} - n\cdot D^{in} = 0 \quad \Longrightarrow \quad \boxed{\varepsilon \frac{\partial \varphi^{out}}{\partial r} = \frac{\partial \varphi^{in}}{\partial r} \quad (1)}$$

$$E_2^{\parallel} - E_1^{\parallel} = 0 \quad \Longrightarrow \quad \boxed{\frac{\partial \varphi^{out}}{\partial \theta} = \frac{\partial \varphi^{in}}{\partial \theta} \quad (2)}$$

In general these equations are easily satified for $l \neq 1$. Just set $B_l = C_l = 0$. The $l=1$ equation gives a non-trival condition

From (1) and (2) we find:

$$\left(-\varepsilon E_0 - 2\varepsilon \frac{B_1}{a^3}\right) = C_1$$

$$\left(-E_0 a + \frac{B_1}{a^2}\right) = C_1 a$$

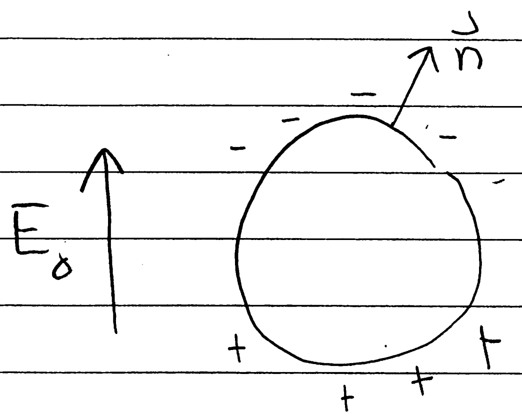
Solving for C_1 and B_1 we have: ~~$(1-\varepsilon)$ is negative~~

$$\varphi = \begin{cases} -E_0 r\cos\theta + \dfrac{E_0 a^3}{r^2}\dfrac{(1-\varepsilon)}{(1+2\varepsilon)}\cos\theta & r>a \\[2ex] -E_0 r\cos\theta \left(\dfrac{3\varepsilon}{1+2\varepsilon}\right) & r<a \end{cases}$$

We make the following remarks:

1. The electric field is constant in the sphere

2. The induced charge is:



$$\sigma_p = -\vec{n}\cdot\left(\vec{P}_2 - \cancel{\vec{P}_1}\right)$$

$$= -\chi\, \vec{n}\cdot\vec{E}$$

$$= -(\varepsilon - 1)\left(-\frac{\partial \varphi^{out}}{\partial r}\right)\Bigg|_{r=a}$$

This yields after differentiating φ^{out}

$$\boxed{\sigma_p = -\frac{3(\varepsilon-1)E_0\cos\theta}{1+2\varepsilon}}$$

thus we find the appropriate sign.