Electrodynamics in media

$$\nabla \cdot E = \rho$$
$$\nabla \times B = j/c + 1/c\, \partial_t E$$

$$\nabla \cdot B = 0$$
$$-\nabla \times E = 1/c\, \partial_t \vec{B}$$

What is the current in the material. The charge is known once $\vec{j}$ is specified.

$$\partial_t \rho + \nabla \cdot \vec{j} = 0$$

Usually divide the charges and currents into external (those explicitly specified) and material currents (those part of material)

$$\vec{j} = \vec{j}_{mat} + \vec{j}_{ext}$$

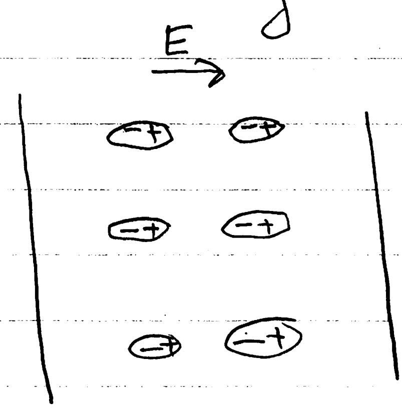
$$\rho = \rho_{mat} + \rho_{ext}$$

Need to specify a constitutive eqn $\vec{j}_{mat}[E]$

Need to also specify the material currents inside the medium before we know the equations to solve

Symmetry is key !

For insulating materials we have a basic picture



The electric field polarizes the material leading to a net dipole moment/volume proportional to the applied field

Constituent Relations:

Now we need to specify $\vec{j}_{mat}$. Treat j_{mat} as an expansion in $\vec{E}$ field and its derivatives

- Electric field is weak keep only first order in E
- Also assume isotropic medium (no preferred axis)

$$\underbrace{\vec{j}_{mat}}_{T\ \text{odd}} = \underbrace{\sigma}_{\substack{\text{conductivity} \\ T\text{-odd}}} \overbrace{\vec{E}}^{T\text{-even}}$$

T-odd ⇐ Dissipative process

For an insulator the conductivity is vanishingly small. We also will consider more terms

$$j^i = \underbrace{\cancel{\sigma}^{\,0} E^i}_{T\text{-odd}} + \underbrace{\chi\, \partial_t E^i}_{T\text{-even}} + \underbrace{\sigma_2\, \partial_t \partial_t E^i}_{T\text{-odd}} + \underbrace{\chi_3\, \partial_t^3 E^i}_{T\text{even}} + \dots$$

When the macroscopic time scales are long compared to the microscopic times, each higher term is suppressed.

Constituent Relations (Continued)

Reason. Dimensional Analysis:

$$[j] = \frac{q}{m^2}\frac{1}{S}$$ S = seconds

m = meter

$$[E] = \frac{q}{m^2}$$ q = charge

So:

Thus expect:

$[\chi] = 1$ $\chi \sim 1$

$[\sigma_2] = S$ $\sigma_2 \sim \tau_{micro}$ (or even less)

$[\chi_3] = S^2$ $\chi_3 \sim \tau_{micro}^2$

$[\sigma_4] = S^3$ $\sigma_4 \sim \tau_{micro}^3$

⋮

While for a macro-time scale $T \gg \tau_{micro}$:

$$\partial_t E \sim \frac{1}{T} E \quad \text{and} \quad \partial_t^2 E \sim \frac{1}{T^2} E \quad \ldots$$

Thus:

$$\vec{j} = \cancel{\sigma \vec{E}}^{\,0} + \chi \partial_t E + \sigma_2 \partial_t^2 E + \chi_3 \partial_t^3 E$$

$$\sim 0 + \frac{E}{T} + \left(\frac{\tau_{mic}}{T}\right)\frac{E}{T} + \left(\frac{\tau_{mic}}{T}\right)^2 \frac{E}{T} + \ldots$$

Each higher term is suppressed by $\left(\frac{\tau_{micro}}{T}\right)$

Constituent Relation (Final)

Thus at lowest order in the gradient expansion

$$\vec{j} = \chi \partial_t \vec{E}$$

$$\boxed{\vec{j} = \partial_t \vec{P}}$$

polarization vector

$$\boxed{\vec{P} = \chi \vec{E}}$$ ← Linear isotropic media

T-even

$\vec{j}$ is T-odd

T-even

So we can work out the charge density:

- From

$$\rho(\omega, k) = \frac{\vec{k} \cdot \vec{J}}{\omega} \quad \text{and} \quad \vec{j}(\omega, k) = -i\omega \vec{P} \Longleftrightarrow \vec{j} = \partial_t \vec{P}$$

Find $\rho(\omega, k) = -i\vec{k} \cdot \vec{P}$ or $\boxed{\rho = -\vec{\nabla} \cdot \vec{P}}$

- Or could have used coordinate space:

$$\partial_t \rho + \nabla \cdot j = 0$$

$$\partial_t \rho + \partial_i \partial_t P^i = 0$$

$$\partial_t (\rho + \partial_i P^i) = 0 \implies \boxed{\rho = -\partial_i P^i}$$

Constituent Relation in EOM

With this we get the Eqs of motion:

$$\nabla \cdot E = \rho_{mat} + \rho_{ext} \qquad \nabla \times E = 0$$

$$\nabla \cdot E = -\nabla \cdot P + \rho_{ext}$$

Now

$$\nabla \cdot (\vec{E} + \vec{P}) = \rho_{ext}$$

$$\nabla \times E = 0$$

with $\vec{P} = \chi \vec{E}$ for linear isotropic matter

So define $\boxed{\vec{D} = \vec{E} + \vec{P}}$ and find

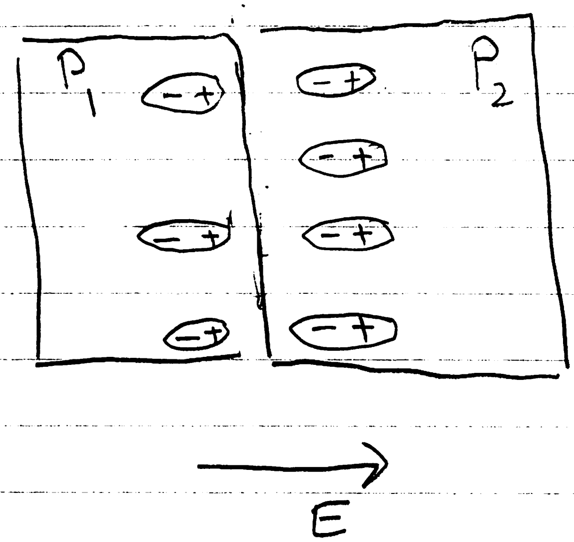
$$\boxed{\left.\begin{aligned} \nabla \cdot \vec{D} &= \rho_{ext} \\ \nabla \times E &= 0 \end{aligned}\right\} \text{eqs of macroscopic matter}}$$

Where $\vec{D} \equiv \vec{E} + \vec{P} \Rightarrow \underbrace{(1+\chi)}_{\equiv \varepsilon} \vec{E}$ for a linear medium isotropic medium

linear relation

$$\varepsilon = 1 + \chi$$

The material charge at the interface



$P_2 > P_1$.

There is a net negative charge at the interface from the material

First lets caclute the surface charge. For simplicity, set the external or "free" charge to zero at the interface:

We showed generally that | for simplicity

$$\vec{n}\cdot(\vec{E}_2-\vec{E}_1) = \sigma = \cancel{\sigma_{ext}}^{\,0} + \sigma_{mat}$$

Then we have

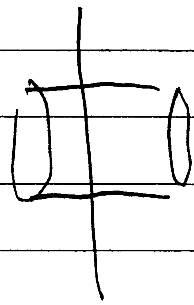
0 simplicity

$$\nabla\cdot E = \cancel{\rho_{ext}} + \rho_{mat}$$

$$\nabla\cdot E = -\nabla\cdot P$$

we showed this on the previous page

So from gauss Law



$$\vec{n}\cdot(\vec{E}_2-\vec{E}_1) = -\vec{n}\cdot(\vec{P}_2-\vec{P}_1)$$

$$\cancel{\vec{\sigma}_{ext}}^{\,0} + \sigma_{mat} = -n\cdot(\vec{P}_2-\vec{P}_1)$$

for simplicity

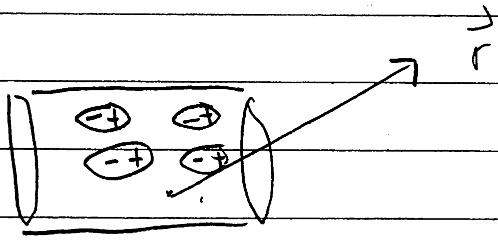
So

$$\boxed{\sigma_{mat} = -\vec{n}\cdot(\vec{P}_2-\vec{P}_1)}$$

This is what we expect based on the dipole picture

Relation to the Dipole Picture (see also Jackson 4,3)

Consider a polarized object, and lets determine the potential at $\vec{r}$.



One could hope that the potential is given by a sum of dipole potentials.

The potential at $\vec{r}$ is

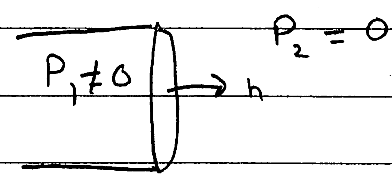
$$\varphi(\vec{r}) = \int_V d^3r_0 \, \frac{\rho_{mat}(r_0)}{4\pi|\vec{r}-\vec{r}_0|} + \int_S da \, \frac{\sigma_{mat}(x)}{4\pi|\vec{r}-\vec{x}|}$$

where $\vec{r}_0$ runs over the volume and $\vec{x}$ runs over the surface

using $\rho = -\nabla \cdot \vec{P}$ and

and $\sigma = -\vec{n}\cdot(\cancel{\vec{P}_2}^{\,0} - \vec{P}_1)$

$= \vec{n}\cdot\vec{P}_1$



Find

$$\varphi(r) = \int d^3r_0 \, \frac{-\partial_i P^i(r_0)}{4\pi|\vec{r}-\vec{r}_0|} + \int_S \frac{d\vec{a}\cdot\vec{P}}{4\pi|\vec{r}-\vec{x}|}$$

integrate by parts:

$$\frac{\frac{\partial P^i}{\partial r_0^i}}{4\pi|\vec{r}-\vec{r}_0|} = \frac{\partial}{\partial r_0^i}\left(\frac{-P^i}{4\pi|\vec{r}-\vec{r}_0|}\right) + \frac{P^i(\vec{r}-\vec{r}_0)_i}{4\pi|\vec{r}-\vec{r}_0|^3}$$

Leading to:

$$\varphi(\vec{r}) = \int d^3r_0 \, \frac{\vec{P}\cdot(\vec{r}-\vec{r}_0)}{4\pi|\vec{r}-\vec{r}_0|^3} \cancel{- \int_X \frac{d\vec{a}\cdot\vec{P}}{4\pi|\vec{r}-\vec{x}|} + \int_X \frac{d\vec{a}\cdot\vec{P}}{4\pi|\vec{r}-\vec{x}|}}$$

This has the form of a dipole field from each volume element