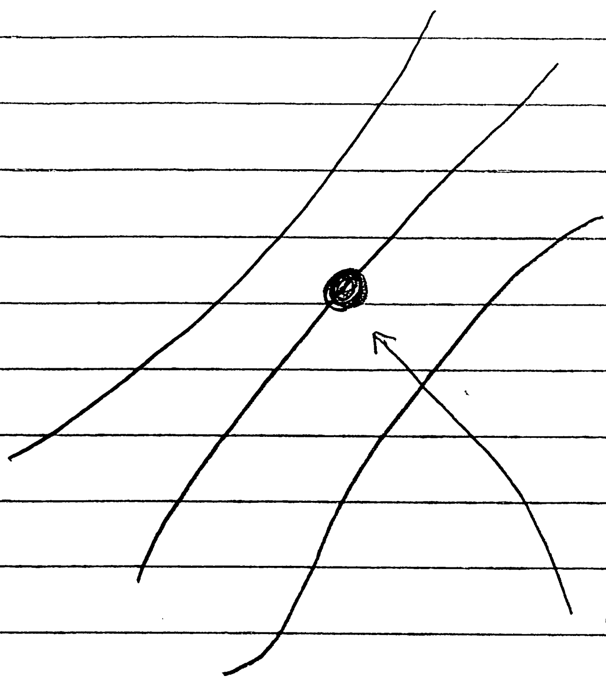
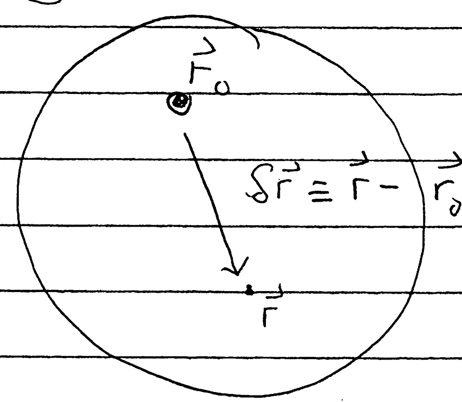
Forces on Small Charged Objects - Multipole



Consider the interaction energy between a <u>small</u> object and an external field, $\vec{E}_{ext}(r_0)$

$\varphi_{ext}(r_0)$ satisfies $-\nabla^2\varphi = 0$

Magnify this picture:



The interaction energy is

$$U_{int} = \int d^3\vec{r}\; \rho(\vec{r})\, \varphi_{ext}(\vec{r})$$

Then we expand the potential around $\vec{r}_0$. Here $\vec{r}_0$ is any fixed point on the body

$$\varphi(r) = \varphi(r_0) + \frac{\partial\varphi(r_0)}{\partial r_0^i}\,\delta r_0^i + \frac{1}{2}\frac{\partial\varphi(r_0)}{\partial r_0^i\,\partial r_0^j}(\delta r_0^i\,\delta r_0^j) + \ldots$$

Use

$$-\nabla^2\varphi = 0 \quad \text{or} \quad \delta^{ij}\frac{\partial}{\partial r_0^i}\frac{\partial}{\partial r_0^j}\varphi(r_0) = 0$$

So write

$$\varphi(r) = \varphi(r_0) + \frac{\partial \varphi}{\partial r_0^i}\,\delta r_0^i + \frac{1}{2}\frac{\partial^2 \varphi}{\partial r_0^i \partial r_0^j}\left(\delta r^i \delta r^j - \frac{1}{3}\delta^{ij}\,\delta \vec{r}^{\,2}\right)$$

So find

$$U_{int} = \varphi(r_0)\, Q_{TOT} + \frac{\partial \varphi}{\partial r_0^i}\,\vec{p}^{\,i} + \frac{1}{6}\frac{\partial^2 \varphi}{\partial r_0^i \partial r_0^j}\, Q^{ij}$$

Where

$$Q_{TOT} \equiv \int_r \rho(\vec{r})$$

$$\vec{p} \equiv \int_r \rho(r)\,\delta\vec{r}$$

$$Q^{ij} \equiv \int_r \rho(r)\left(3\,\delta r^i \delta r^j - \delta^{ij}\,\delta r^2\right)$$

Or

$$\boxed{U_{int}(r_0) = Q_{TOT}\,\varphi(r_0) - \vec{p}\cdot\vec{E}(r_0) - \frac{1}{6}\,Q^{ij}\,\partial_i E_j(\vec{r}_0)} \quad + \ldots$$

To find the force we have

$$\vec{F} = -\nabla_{r_0} U_{int}(r_0)$$

You will need this for the homework