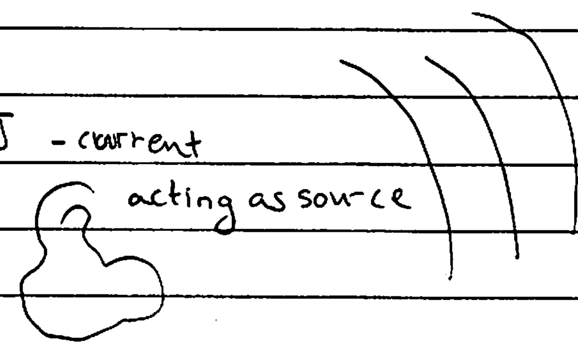
Green Fcn of the Wave-Eqn

$$-\Box u(t,x) = J(t,x)$$

source. In E+M these will be currents

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$



The induced waves are

$$u(t,x) = \int dt_0\, dx_0\, G_R(t-t_0, \vec{x}-\vec{x}_0)\, J(t_0,x_0)$$

Then $G_R(t\vec{x}\,|\,t_0 x_0)$ is the field at $t\vec{x}$ due to a point charge at $t_0 x_0$

$$-\Box G_R(t,\vec{x}\,|\,t_0 x_0) = \delta(t-t_0)\,\delta^3(x-x_0)$$

Then:

$$-\Box u(t,x) = \int \overbrace{-\Box G_R(tx\,|\,t_0 x_0)}^{\delta(t-t_0)\delta^3(x-x_0)}\, J(t_0,x_0)$$

$$-\Box u(t,x) = J(t,x) \quad \checkmark$$

Solving for the Grn-fcn

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(t-t_0, \vec{x}-\vec{x}_0) = \delta(t-t_0)\,\delta^3(\vec{x}-\vec{x}_0)$$

First choose $t_0, x_0 = 0$;

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] G(t,\vec{x}) = \delta(t)\,\delta^3(\vec{x})$$

Now Fourier transform in Space: $G(t,\vec{k}) = \int e^{-i\vec{k}\cdot\vec{x}}\, G(t,x)$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + k^2\right] G(t,k) = \delta(t)$$

or

$$\frac{1}{c^2}\left[\frac{\partial^2}{\partial t^2} + (ck)^2\right] G(t,k) = \delta(t)$$

Compare to SHO:

$$m\left[\frac{\partial^2}{\partial t^2} + \omega_0^2\right] G(t) = \delta(t)$$

Thus we can take the SHO result "lock-stock + barrel" with $\omega_0 = ck$ and $m \to 1/c^2$

$$G(\tau,k) = c^2\,\Theta(\tau)\,\frac{\sin(ck\tau_0)}{ck}$$

Now we "only" need to take the inverse FT:

$$G(\tau, \vec{r}) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k}\cdot\vec{r}} \, c^2 \Theta(\tau) \frac{\sin ck\tau}{ck}$$

The integral is not convergent. But this is not a surprise. Add a convergence factor

$$G_\varepsilon(t, \vec{r}) = \int \frac{d^3k}{(2\pi)^3} \, e^{-\varepsilon|\vec{k}|} \, e^{i\vec{k}\cdot\vec{r}} \, c^2 \Theta(\tau) \frac{\sin ck\tau}{ck}$$

Think of $G(t, \vec{r})$ as a limit of sequence, $G_\varepsilon(t, \vec{r})$, of functions which satisfy

A dirac sequence

$$-\Box G_\varepsilon(t, r) = \delta(t)\, \delta^3_\varepsilon(\vec{r})$$

$\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} e^{-\varepsilon|k|}$

To do the integral write $R = |\vec{r}|$

$$G_\varepsilon(t, \vec{r}) = \int \frac{k^2 dk \; d(\cos\theta)\, d\phi}{(2\pi)^3} \, e^{-\varepsilon k} \, e^{ikR\cos\theta} \, c^2 \Theta(\tau) \frac{\sin ck\tau}{ck}$$

Doing angular integral $\int_{-1}^{1} d(\cos\theta)\, e^{ikR\cos\theta} = \frac{2\sin kR}{kR}$

we find

$$G_\varepsilon(t, \vec{r}) = \frac{1}{2\pi^2} \frac{c\Theta(\tau)}{R} \int_0^\infty e^{-\varepsilon k} \sin kR \, \sin ck\tau$$

The remaining integrals can be done writing

$$\sin kR \sin ck\tau = \frac{1}{2}\left[\cos(k(R-c\tau)) + \cos(k(R+c\tau))\right]$$

and with $\cos k(R-c\tau) = \frac{1}{2}\left[e^{ik(R-c\tau)} + e^{-ik(R-c\tau)}\right]$

So that

$$\int_0^\infty dk\, e^{-\varepsilon k}\cos(k(R-c\tau)) = \frac{\varepsilon}{\left[(R-c\tau)^2+\varepsilon^2\right]}$$

We find

$$G_\varepsilon = \frac{1}{4\pi R}\, c\,\Theta(\tau)\left[\frac{1}{\pi}\frac{\varepsilon}{(R-c\tau)^2+\varepsilon^2} + \frac{1}{\pi}\frac{\varepsilon}{(R+c\tau)^2+\varepsilon^2}\right]$$

Using

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} = \delta(x)$$

Find

can't be satisfied $R>0,\ \tau>0$

$$G(\tau,R) = \frac{c}{4\pi R}\,\Theta(\tau)\left[\underbrace{\delta(R-c\tau)}_{\substack{\text{pull out } c \\ \frac{1}{c}\delta\left(\frac{R}{c}-\tau\right)}} + \delta(R+c\tau)\right]$$

$$\boxed{G(\tau,R) = \frac{1}{4\pi R}\,\Theta(\tau)\,\delta\left(\frac{R}{c}-\tau\right)}$$

More generally

$$G(t-t_0, \vec{r}-\vec{r}_0) = \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\,\theta(t-t_0)\,\delta\left(\frac{|\vec{r}-\vec{r}_0|}{c} - (t-t_0)\right)$$

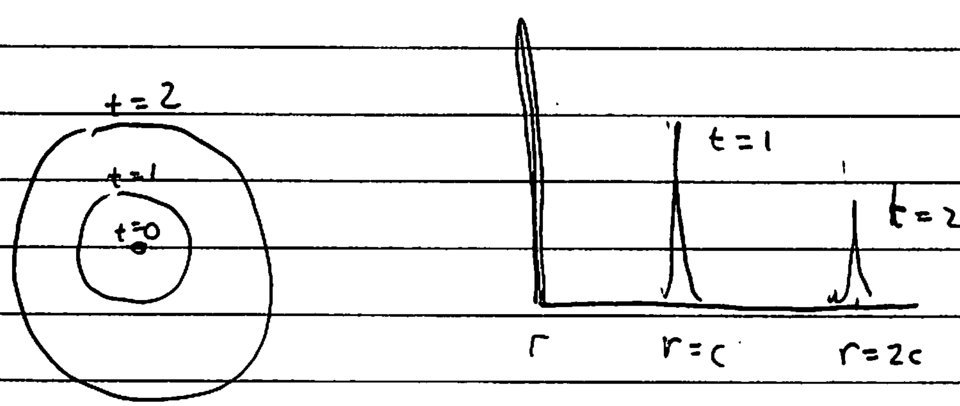
So

$$u(t,x) = \int dt_0\, d^3\vec{r}_0\; G(t-t_0, \vec{r}-\vec{r}_0)\, J(t_0, \vec{r}_0)$$

$$\boxed{u(t,x) = \int d^3\vec{r}_0\, \frac{1}{4\pi|\vec{r}-\vec{r}_0|}\, J\left(t - \frac{|\vec{r}-\vec{r}_0|}{c}, \vec{r}_0\right)}$$

retarded time

Picture:



Also record

$$\boxed{G_R(\omega,k) = \frac{c^2}{\left(-(\omega+i\epsilon)^2 + (ck)^2\right)}}$$

Summary

$$G_R(\tau, R) = \frac{\Theta(\tau)}{4\pi R}\,\delta\!\left(\frac{R}{c} - \tau\right)$$

$$G_R(\tau, \vec{k}) = c^2\,\Theta(\tau)\,\frac{\sin(ck\tau)}{ck}$$

$$G_R(\omega, k) = \frac{c^2}{\left[-(\omega + i\varepsilon)^2 + (ck)^2\right]}$$