Problem 1. Proper acceleration

A particle of mass m, starting at rest at time $t = 0$ and $x = 0$ in the lab frame, experiences a constant acceleration, a, in the x-direction in its own rest frame.

(a) The acceleration four vector

$$A^\mu \equiv \frac{d^2 x^\mu}{d\tau^2} \tag{1}$$

is specified by the problem statement. What are the four components of the acceleration four vector in the rest frame of the particle and in the lab frame. What is the acceleration, d^2x/dt^2, in the lab frame.

(b) Show that the position of the particle as a function of time can be parameterized by a real number p

$$x = \frac{c^2}{a}\left[\cosh(p) - 1\right] \tag{2}$$

where p is related to the time t through the equation:

$$c\,t = \frac{c^2}{a}\sinh(p) \tag{3}$$

(c) Show that the parameter p is proportional proper time, $p = \frac{a}{c}\tau$.

(d) The rapidity of a particle, Y, is defined by its velocity

$$\frac{v}{c} \equiv \tanh(Y) \tag{4}$$

where $v = dx/dt$. Show that the four velocity $u^\mu = dx^\mu/d\tau$ is related to the rapidity through the hyperbolic relations.

$$(u^0/c, u^1/c) = (\cosh(Y), \sinh(Y)) \tag{5}$$

(e) Show that $Y = a\tau/c$

Remark: We see that the rapidity of the particle increases linearly with proper time during proper acceleration.

(f) (Optional) If the particle has a constant decay rate in its own frame of Γ, show that the probability that the particle survives at late time t is approximately

$$\left(\frac{2at}{c}\right)^{-\Gamma c/a}$$

Problem 2. The stress tensor from the equations of motion

In class we wrote down energy and momentum conservation in the form

$$\frac{\partial \Theta^{\mu\nu}_{\rm mech}}{\partial x^\mu} = F^{\nu}_{\ \rho}\frac{J^\rho}{c} \tag{6}$$

Where the $\nu = 0$ component of this equation reflects the work done by the E&M field on the mechanical constituents, and the spatial components ($\nu = 1, 2, 3$) of this equation reflect the force by the E&M field on the mechanical constituents.

(a) Verify that

$$F^{\nu}_{\ \rho}\frac{J^\rho}{c} = \begin{cases} \boldsymbol{J}/c \cdot \boldsymbol{E} & \nu = 0 \\ \rho E^j + (\boldsymbol{J}/c \times \boldsymbol{B})^j & \nu = j \end{cases} \tag{7}$$

(b) (**Optional**) Working within the limitations of magnetostatics where

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{J}}{c} \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{8}$$

show that the magnetic force can be written as the divergence of the magnetic stress tensor, $T^{ij}_B = -B^iB^j + \frac{1}{2}\delta^{ij}B^2$:

$$(\frac{\boldsymbol{J}}{c} \times \boldsymbol{B})^j = -\partial_i T^{ij}_B \tag{9}$$

(c) Consider a solenoid of infinite length carrying current I with n turns per length, what is the force per area on the sides of the solenoid. Use the magnetic stress tensor of part (b).

(d) Using the equations of motion in covariant form

$$-\partial_\mu F^{\mu\rho} = \frac{J^\rho}{c} \tag{10}$$

and the Bianchi Identity

$$\partial_\mu F_{\sigma\rho} + \partial_\sigma F_{\rho\mu} + \partial_\rho F_{\mu\sigma} = 0 \tag{11}$$

show that

$$F^{\nu}_{\ \rho}\frac{J^\rho}{c} = -\frac{\partial}{\partial x^\mu}\Theta^{\mu\nu}_{\rm em} \tag{12}$$

where

$$\Theta^{\mu\nu}_{\rm em} = F^{\mu\rho}F^{\nu}_{\ \rho} + g^{\mu\nu}\left(-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{13}$$

Hint: use the fact that $F^{\mu\rho}$ is anti-symmetric under interchange of μ and ρ.

(e) (**Optional**) Verify by direct substitution, using $F^{ij} = \epsilon^{ijk}B_k$, that if there is no electric field that

$$\Theta^{ij}_{\rm em} = T^{ij}_B = -B^iB^j + \frac{1}{2}B^2\delta^{ij}\,. \tag{14}$$

Problem 3. Fields from moving particle

The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$ were given in class. Choose the axes so that the charge moves along the $z-$axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define a transverse vector (or impact parameter) $\boldsymbol{b}_\perp = (x, y)$, with components x and y. Consider the fields and the source in the limit $\beta \to 1$

(a) First (keeping β finite) find the vector potential A^μ associated with the moving particle using a Lorentz transformation. Determine the field strengh tensor $F^{\mu\nu}$ by differentiating A^μ, and verify that you get the same answer as we got in class.

(b) As the charge q passes by a charge e at impact parameter $\boldsymbol{b}$, show that the accumulated transverse momentum transfer (transverse impulse) to the charge e during the passage of q is

$$\Delta \boldsymbol{p}_\perp = \frac{eq}{2\pi} \frac{\boldsymbol{b}_\perp}{b_\perp^2 c} \tag{15}$$

(c) Show that the time integral of the absolute value of the longitudinal force to a charge e at rest at an impact parameter $b_\perp$ is

$$\frac{eq}{2\pi\gamma b_\perp c} \tag{16}$$

and hence approaches zero as $\beta \to 1$.

(d) (**Optional**) Show that the fields of charge q can be written for $\beta \to 1$ as

$$\boldsymbol{E} = \frac{q}{2\pi} \frac{\boldsymbol{b}_\perp}{b_\perp^2} \delta(ct - z)\,, \qquad \boldsymbol{B} = \frac{q}{2\pi} \frac{\hat{\boldsymbol{v}}/c \times \boldsymbol{b}_\perp}{b_\perp^2} \delta(ct - z)\,. \tag{17}$$

(e) (**Optional**) Show by explicit substitution into the Maxwell equations that these fields are consistent with the 4$-$vector source density

$$J^\alpha = q v^\alpha \delta^2(\boldsymbol{b}_\perp) \delta(ct - z) \tag{18}$$

where $v^\alpha = (c, \hat{\boldsymbol{v}})$.

Problem 4. The Relativistic Hamiltonian

Recall that the action of a relativistic point particle is

$$S = -mc^2 \int d\tau + \frac{e}{c} \int A_\mu(X) dX^\mu \,, \tag{19}$$

where the trajectory $X^\mu(\lambda)$ is parameterized by the real number λ, and we use the short hand notation

$$c\, d\tau \equiv \sqrt{-\frac{dX^\mu}{d\lambda}\frac{dX_\mu}{d\lambda}}\, d\lambda \,, \qquad dX^\mu \equiv \frac{dX^\mu}{d\lambda} d\lambda \,. \tag{20}$$

(a) Parameterize the path by time by taking $\lambda = t$ and thus

$$X^\mu(t) = (c\,t, \boldsymbol{x}(t)) \,. \tag{21}$$

Explicitly write down the action $S[\boldsymbol{x}(t)]$ in terms of $\boldsymbol{x}(t)$ and the familiar scalar and vector potentials $(\Phi(t, \boldsymbol{x}), \boldsymbol{A}(t, \boldsymbol{x}))$.

(b) Expand $S[\boldsymbol{x}(t)]$ from part (a) for non-relativistic motion, and verify that the Euler-Lagrange equations lead to the correct non-relativistic equations of motion, with the appropriate Lorentz force $\boldsymbol{F} = e(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B})$.

(c) Use the non-relativistic action of part (b) to determine the canonical momentum, $\boldsymbol{p}_{\rm can} = \partial L/\partial \dot{\boldsymbol{x}}$ and the Hamiltonian ($H = \boldsymbol{p}_{\rm can} \cdot \boldsymbol{x} - L$) in the non-relativistic limit. This is the form normaly used in quantum mechanics.

(d) Now use the relativistic action $S[\boldsymbol{x}(t)]$ (i.e. parameterized by time as in part (a)) to determine the canonical momentum $\boldsymbol{p}_{\rm can}$ and the relativistic Hamiltonian. You should find

$$H = c\sqrt{(\boldsymbol{p}_{\rm can} - \frac{e}{c}\boldsymbol{A})^2 + (mc)^2} + e\,\Phi(t, \boldsymbol{x}) \tag{22}$$

Problem 5. Moving conductors

The constitutive relation is a relation between the macroscopic electrical current density in a medium and the applied fields. Recall that for a normal isotropic conductor *at rest* in an electric ($\boldsymbol{E}$) and magnetic field ($\boldsymbol{B}$) the constitutive relation in a linear response approximation is known as Ohm's Law:

$$\boldsymbol{J} = \sigma \boldsymbol{E} \,. \tag{23}$$

The conductor is uncharged in its rest frame, but has a non-zero charge density in other frames.

(a) By making a Lorentz transformation of the current and fields for small boost velocities:

(i) Deduce the familiar constitutive relation[1] for a normal conductor *moving* non-relativistically with velocity $\boldsymbol{v}$ in an electric and magnetic field from the rest frame constitutive relation, Eq. (23). Iterpret the result in terms of the lorentz force.

(ii) Show that the charge density in the moving conductor is $\rho \simeq \boldsymbol{v} \cdot \boldsymbol{J}/c$. Under what conditions is the charge density positive or negative? Does a loop of wire, which in its rest frame is uncharged and carries a current I, remain uncharged when it is moving with velocity $\boldsymbol{v}$? Explain.

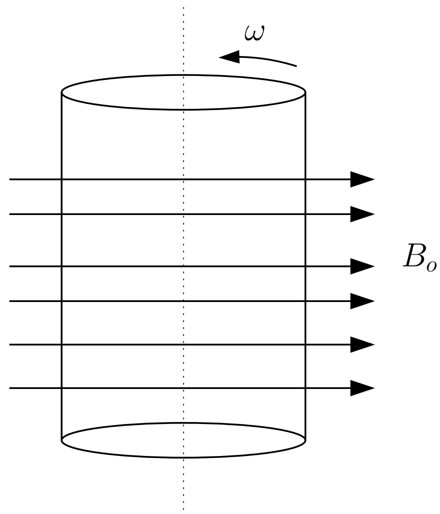
(b) In a general Lorentz frame the conductor moves with four velocity U^μ (here $U^\mu = (c, \mathbf{0})$ in the conductors rest frame, and $U^\mu = (\gamma c, \gamma \boldsymbol{v})$ in other frames). The constitutive relation in Eq. (23) can be expressed covariantly as

$$J^\mu = \frac{\sigma}{c} F^{\mu\nu} u_\nu \tag{24}$$

(i) Check that Eq. (24) reproduces the current and charge density of part (a) in the small velocity limit, $v \ll c$.

(c) Now consider a solid conducting cylinder of radius R and conductivity σ rotating rather slowly with constant angular velocity ω in a uniform magnetic field B_o perpendicular to the axis of the cylinder as shown below. Determine the current flowing in the cylinder and sketch the result.

[1] $\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v}/c \times \boldsymbol{B})$



(d) Determine the torque required to maintain the cylinder's constant angular velocity. Assume that the skin depth is much larger than the radius of the cylinder.

(e) (**Optional**) Evaluate the current numerically (in Amps) for a typical strong laboratory field $\sim 1T$, and rotation frequency $\sim 1\,\mathrm{Hz}$, for Cu wire of radius $\sim 1\,\mathrm{cm}$.