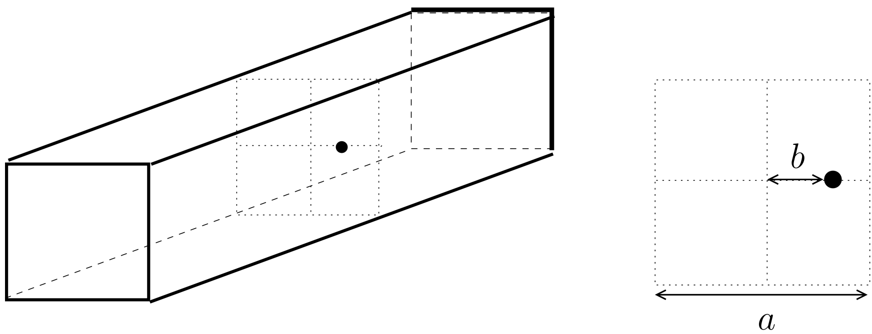
Do not hand in optional parts!

Problem 1. A point charge in a rectangular tube

Consider a point charge placed in an infinitely long grounded rectangular tube as shown below. The sides of the square cross sectional area of the tube have length a.



(a) (Optional) Show that the solutions to the *homogeneous* Laplace equation (i.e. without the extra point charge) are linear combinations of functions of the form

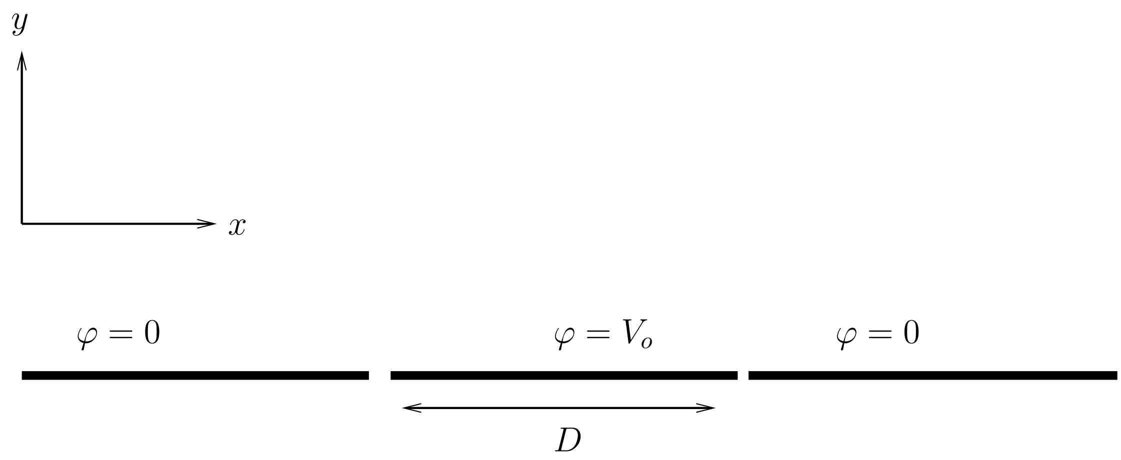
$$\Phi(k_x z)\,\Phi(k_y y)\,e^{\pm\kappa_z z} \qquad \text{where} \qquad \Phi(u) = \Big\{\cos(u) \text{ or } \sin(u) \tag{1}$$

for specific values of k_x, k_y and κ_z. Determine the allowed the values of k_x, k_y and κ_z and their associated functions.

(b) (Optional) Now consider a point charge displaced from the center of the tube by a distance b in the x direction, i.e. the coordinatess of the charge are $\boldsymbol{r}_o = (x, y, z) = (b, 0, 0)$. Use the method of images to determine the potential.

(c) As an alternative to the method of images, use a series expansion in terms of the homogeneous solutions of part (a) to determine the potential from the point charge described in part (b).

(d) Determine the asymptotic form of the surface charge density, and the force per area on the walls of the rectangular tube far from the point charge, i.e. $z \gg a$.



Problem 2. Potential from a strip.

An infinite conducting strip of width D (between $0 < x < D$) is maintained at potential V_o, while on either side of the strip are grounded conducting planes. The strip and the planes are separated by a tiny gap as shown below.

(a) Following a similar example given in class, determine the potential everywhere in the upper half plane $y > 0$.

(b) Determine the surface charge density on the strip and on the grounded planes, and make a graph.

Problem 3. Force between two rings of charge

A single ring of charge of radius a and total charge Q is centered at the origin and lies in the xy plane.

(a) Show that the potential far from the ring can be written as the multipole expansion

$$\Phi = \frac{Q}{4\pi} \sum_{\ell} \frac{a^{\ell} P_{\ell}(0)}{r^{\ell+1}} P_{\ell}(\cos\theta) \tag{2}$$

$$\simeq \frac{Q}{4\pi r} - \frac{1}{2}\frac{Qa^2}{4\pi r^3} P_2(\cos\theta) + \frac{3}{8}\frac{Qa^4}{4\pi r^5} P_4(\cos\theta) + \ldots \tag{3}$$

where θ is measured relative to the z axis, and were in the second line we have used the known values for $P_{\ell}(0)$. What are the values of the spherical multipoles $q_{\ell m}$ (up to $\ell = 2$), and the cartesian multipoles p_i and $\mathcal{Q}_{ij}$.

(b) For a ring of charge of radius a, use an elementary argument to determine the potential along the z axis. Verify that it agrees with the expansion of part (a) when part (a) is evaluated on the z axis.

(c) Show that the force between two coaxial charged rings of charge Q and $-Q$ widely separated by a distance, $2Z$, along the z axis is

$$F \simeq \frac{-Q^2}{16\pi Z^2} + 3\frac{Q^2 a^2}{64\pi Z^4} + \ldots \tag{4}$$

where a negative answer indicates an attractive force.

An elegant way to find this is to use the Green Reciprocity theorem (read sec 3.5.2), which in this context says that the potential energy of a quadrupole charge distribution in the electrostatic potential from a monopole is the same as the potential energy of a monopole in an electrostatic potential from a quadrupole.

Problem 4. A ring of charge close to a plane

(a) Consider a long *line* of charge separated from a grounded plane by separation z_o. The charge per length is λ. Determine the force per length between the grounded plane and the charged line.

(b) By integrating the force found in part(a), show that the potential energy per length of the line of charge and the grounded plane is

$$u_{\rm int} = \frac{\lambda^2}{4\pi} \log 2z_o + \text{const} \tag{5}$$

This potential energy for is exactly half of the potential energy between the line of charge and its image. Qualitatively explain why this is the case.

(c) Conser a ring of radius a and total charge Q, separated from a plane by a height z_o. Use the results of this problem to determine the total force between the ring and the plane when $z_o \ll a$. Explain qualitatively why the results of this problem apply.

Problem 5. The free green function in cylindrical coordinates

(a) Show that the green function in cylindrical coordinates can be expanded as

$$G(\boldsymbol{r}, \boldsymbol{r}_o) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty k dk \, \left[e^{im\phi} J_m(k\rho)\right] \left[e^{-im\phi_o} J_m(k\rho_o)\right] g_{km}(z, z_o) \tag{6}$$

and determine the appropriate equation for $g_{km}(z, z_o)$. (Hint: this may be a good time to examine the course notes and to write $\delta^3(\boldsymbol{r} - \boldsymbol{r}_o) = \frac{1}{\rho}\delta(\rho - \rho_o)\delta(z - z_o)\delta(\phi - \phi_o)$ as an expansion in eigen functions in the ρ, ϕ directions)

(b) (Optional) If you dont know what a Bessel function looks like, plot $J_0(x)$, $J_1(x)$, $J_2(x)$ and record their series expansions at small and large x. Be aware of the following indentity in 2 dimensions: the 2D function

$$e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp} \tag{7}$$

can be written as a fourier series at each radius $r_\perp$. Defining $\boldsymbol{r}_\perp = r_\perp(\cos\phi, \sin\phi)$, we have

$$e^{ik_\perp r_\perp \cos\phi} = \sum_{m=-\infty}^{\infty} e^{im\phi} \, i^m J_m(kr_\perp) \tag{8}$$

(c) Consider a two dimensional function $f(r_\perp)$ which is independent of the azimuthal angle ϕ. Its Fourier transform, $\hat{f}(k_\perp) = \int d^2\boldsymbol{x}_\perp e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{r}_\perp}$, is independent of the azimuthal of the $\boldsymbol{k}_\perp$. Using Eq. (8), determine an integral relation between $\hat{f}(k_\perp)$ and $f(r_\perp)$ (and vice versa) using the Bessel function $J_0(k_\perp r_\perp)$. We say that $\hat{f}(k_\perp)$ and $f(r_\perp)$ are (up to a constant) Hankel transforms of each other (google Hankel transform).

(d) Use the method of direct integration to show that the free Green function in cylindrical coordinates can be written

$$G_o(\boldsymbol{r}, \boldsymbol{r}_o) \equiv \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty k dk \, \left[e^{im\phi} J_m(k\rho)\right] \left[e^{-im\phi_o} J_m(k\rho_o)\right] \frac{e^{-k(z_> - z_<)}}{2k} \tag{9}$$

where $z_>$ and $z_<$ is the greater and lesser of z and z_o.

It is useful to compare this result to the one derived in class

$$\frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = \sum_{\ell=0}^{\infty} \sum_{-\ell}^{\ell} \left[Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta_o, \phi_o)\right] \frac{1}{2\ell + 1} \frac{r_<^\ell}{r_>^{\ell+1}} \tag{10}$$

and to a similar problem that could have been asked

$$\frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int \frac{dk}{2\pi} \left[e^{im(\phi - \phi_o)} e^{ik(z - z_o)}\right] I_m(k\rho_<) K_m(k\rho_>) \tag{11}$$

Problem 6. The potential energy of a charged ring

A charged ring of radius a and total charge Q is at a height z_o above a grounded plane.

(a) Show that the interaction energy between the plane and the ring is

$$U_{\rm int}(z_o) = U(z_o) - U_{\rm self} = \frac{1}{2}\int_{\rm ring} d^3r \int_{\rm ring} d^3r_1\, \rho(\boldsymbol{r})\left[G(\boldsymbol{r},\boldsymbol{r}_1) - G_o(\boldsymbol{r},\boldsymbol{r}_1)\right]\rho(\boldsymbol{r}_1) \tag{12}$$

where $G(\boldsymbol{r},\boldsymbol{r}_1)$ is the green function of a point charge in the presence of the grounded plane, and $G_o(\boldsymbol{r},\boldsymbol{r}_1)$ is the free green function.

(b) From the image solution for the Green function and the expansion given in Eq. (9), show that the interaction energy of a ring with a grounded potential is

$$U_{\rm int}(z_o) = -\frac{Q^2}{8\pi a}\int_0^\infty dx\, \left[J_0(x)\right]^2 e^{-2x(z_o/a)} \tag{13}$$

The last remaining integral can be done (with Mathematica)

$$U_{\rm int}(z_o) = -\frac{Q^2}{8\pi a}\left[\frac{a}{z\pi}\,\text{EllipticK}(-\frac{a^2}{z^2})\right] \tag{14}$$

(c) Starting from the integral in Eq. (13) and the expansion of the Bessel function (see DLMF), determine the asymptotic form of the force on the ring for $z_o \gg a$. You should find that your result is in agreement with Eq. (4).

(d) Use the series expansions of complete elliptic integrals available in Mathematica (`FullSimplify[Series[EllipticK[-y],...], Assumptions->{y>0}]` worked for me), to show that the potential energy between the ring and the plane is:

$$U_{\rm int}(z_o) \simeq \frac{Q^2}{8\pi^2 a}\log(z_o/4a) \tag{15}$$

Compute the force and verify consistency with Problem 2.

(e) Use Mathematica or other program to plot the potential energy $U_{\rm int}(z_o)/(Q^2/4\pi a)$ versus z_o/a, together with the asymptotics all in one plot.