

Problem 1. A time dependent dipole

Consider an electric dipole at the spatial origin ($\mathbf{x} = 0$) with a time dependent electric dipole moment oriented along the z-axis, *i.e.*

$$\mathbf{p}(t) = p_o \cos(\omega t) \hat{\mathbf{z}}, \quad (1)$$

where $\hat{\mathbf{z}}$ is a unit vector in the z direction.

- (a) Recall that the near and far fields of the time dependent dipole are qualitatively different. Estimate the length scale that separates the near and far fields.
- (b) In the far field, how do the magnitude of the field strengths decrease with radius?
- (c) Determine the ratio of E/B at a distance r in the far field.
- (d) Estimate the total power radiated in a dipole approximation. How does this power depend on the dipole amplitude p_o , the oscillation frequency ω , and the speed of light.
- (e) In the near field regime, *estimate* how the electric and magnetic field strengths decrease with the radius r . (r is the distance from the origin to the observation point.)
- (f) *Estimate* the ratio E/B at a distance r in the near field. Is this ratio large or small?
- (g) Determine the electric and magnetic fields to the lowest non-trivial order in the near field (or quasi-static) approximation.

Problem 2. Radiation in the lowest Bohr Orbit

In the Bohr model, a classical non-relativistic electron circles a proton in a circular orbit with angular momentum $L = \hbar$, due to the Coulomb attraction between the electron and the proton.

- (a) **(Optional)** Recall that the electron kinetic energy is half of minus its potential energy (for a coulomb orbit). Recall also that the lowest bohr orbit has velocity, $\beta = \alpha$ where $\beta = v_e/c$, and $\alpha = e^2/(4\pi\hbar c) = 1/137$. Prove these statements.
- (b) Write down the (total=kinetic + potential) energy and radius of the lowest Bohr orbit in terms of the electron mass, m_e , \hbar , c and α . What is the size of the Bohr radius a_o compared to the electron compton wavelength, *i.e.* $a_o/(\hbar/(m_e c))$?
- (c) One of the difficulties with the Bohr model, is that classically the electron would radiate. Determine the energy lost to radiation per unit time, for an electron in the lowest orbit.
- (d) Determine the energy radiated per revolution in the Bohr model, ΔE , and compare ΔE to the (kinetic+potential) energy of the orbit, *i.e.* compute $\Delta E/E_{\text{orbit}}$. Express $\Delta E/E_{\text{orbit}}$ in terms of the fine structure constant, and estimate its value.
- (e) If the electron moves in the x, y plane determine the time averaged power radiated per solid angle, $\overline{dP}/d\Omega$.

You should find

$$\frac{\overline{dP}}{d\Omega} = \frac{e^2}{16\pi^2 c^3} \frac{1}{2} (1 + \cos^2 \theta) (\omega_o^2 a_o)^2 \quad (2)$$

where ω_o is the angular velocity of the electron

- (f) Check your result of part (e) by integrating over solid angle and comparing with part (c).
- (g) Now we will study the polarization of the light. (These questions do not require calculation).
 - (i) If the emitted light is observed along x axis, what is the polarization of the radiated light? Explain physically.
 - (ii) If the emitted light is observed along the y axis, what is the polarization of the radiated light? Explain physically.
 - (iii) If the emitted light is observed along the z axis, what is the polarization of the light? Explain physically.
- (h) The power radiated along the z -axis is twice as large as the power radiated along the x -axis. Explain this result physically.

Problem 3. Radiation from a Phased Array

A current distribution consists of N identical sources. The k -th source is identical to the first source except for a rigid translation by an amount \mathbf{R}_k ($k = 1, 2, \dots, N$). The sources oscillate at the same frequency but have different phases δ_k . That is

$$\mathbf{j}_k \propto \exp(-i(\omega t + \delta_k)) \quad (3)$$

- (a) Show that the angular distribution of radiated power can be written as a product of two factors: one is the angular distribution for $N = 1$; the other depends on \mathbf{R}_k and δ_k , but not on the structure of the sources.
- (b) The planes of two square loops (each with sided length a) are centered on (and lie perpendicular to) the z -axis at $z = \pm a/2$. The loop edges are parallel to the x and y coordinate axes. Find the angular distribution of power in the $x - z$ plane if the current at all points in both loops is $I \cos(\omega t)$. Make a polar plot of the angular distribution of power for $\omega c/a = 2\pi$ and $\omega c/a \ll 1$. Identify the multipole character of the radiation in the limit $\omega a/c \ll 1$.

You should find

$$\frac{dP}{d\Omega} = \frac{I_o^2 a^2 \omega^2}{32\pi^2 c^3} (2 \sin(\sin \theta ka/2))^2 (2 \cos(\cos \theta ka/2))^2 \quad (4)$$

- (c) The limit $\omega a/c \ll 1$ has a simple physical interpretation. Describe this interpretation and show that it reproduces all aspects of the power distribution (including normalization factors) in the limit $\omega a/c \ll 1$.
- (d) Repeat part (b) when the current in the upper loop is $I \cos \omega t$ and the current in the lower loop is $-I \cos \omega t$.

Problem 4. A Charged Rotor: Zangwill

Two identical point charges of charge q are fixed to the ends of a rod of length 2ℓ which rotates with constant angular velocity of $\frac{1}{2}\omega$ in the $x - y$ plane about an axis perpendicular to the rod and through its center

- (a) Calculate the electric dipole moment $\mathbf{p}(t)$. Is there electric dipole radiation?
- (b) Calculate the magnetic dipole moment $\mathbf{m}(t)$. Is there magnetic dipole radiation?
- (c) Show that the electric quadrupole moment is

$$Q(t) = 3q\ell^2 \begin{pmatrix} \frac{1}{3} + \cos \omega t & \sin \omega t & 0 \\ \sin \omega t & \frac{1}{3} - \cos \omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \quad (5)$$

- (d) Show that the time averaged angular distribution of radiated power is

$$\frac{\overline{dP}}{d\Omega} = \frac{1}{128\pi^2 c^5} \omega^6 q^2 \ell^4 (1 - \cos^4 \theta). \quad (6)$$

Problem 5. (Almost all optional) Basics of Relativity

Do not hand in optional parts

- (a) (Optional) The space time event at $X^\mu = (X^0, X^i) = (ct, \mathbf{x})$ happens at $\underline{X}^\mu = (\underline{X}^0, \underline{X}^i) = (c\underline{t}, \underline{\mathbf{x}})$ according to an observer moving to the right along the x axis with velocity v . Define the “light-cone” coordinates $x^+ \equiv X^0 + X^1$ and $x^- \equiv X^0 - X^1$. Show that under this boost that the x^+ coordinates are contracted, while the x^- coordinates are elongated

$$\underline{x}^+ = e^{-y} x^+ = \sqrt{\frac{1-\beta}{1+\beta}} x^+, \quad (7)$$

$$\underline{x}^- = e^y x^- = \sqrt{\frac{1+\beta}{1-\beta}} x^-. \quad (8)$$

Here

$$y = \tanh^{-1} \beta = \frac{1}{2} \log \left(\frac{1+\beta}{1-\beta} \right) \quad (9)$$

is the so-called “rapidity” of the boost. What is $\underline{x}^+ \underline{x}^-$ and why is it unchanged under boost?

- (b) (Optional) A Lorentz tensor transforms as

$$\underline{T}^{\mu\nu} = L^\mu_\rho L^\nu_\sigma T^{\rho\sigma} \quad (10)$$

Show that the transformation rule can be alternatively written

$$\underline{T}^\mu_\nu = (\mathcal{L})^\mu_\rho T^\rho_\sigma (\mathcal{L}^{-1})^\sigma_\nu \quad (11)$$

or equivalently

$$\underline{T}^\mu_\nu = L^\mu_\rho L_\nu^\sigma T^\rho_\sigma \quad (12)$$

- (c) (Optional) The frequency and wave number of a plane wave of light, $e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} = e^{iK\cdot X}$, form a lightlike four vector

$$K^\mu = \left(\frac{\omega}{c}, \mathbf{k} \right) \quad (13)$$

- (i) Show that $K \cdot K = K_\mu K^\mu = 0$ (this is the statement that K is lightlike.)
(ii) If a photon has frequency ω_o and is propagating along the z -axis, show (using the 4-vector properties of K^μ) that according to an observer propagating in the negative z direction with speed β

$$\omega = \sqrt{\frac{1+\beta}{1-\beta}} \omega_o \quad (14)$$

- (d) (Optional) Show that the four velocity $U^\mu = dx^\mu/d\tau$ satisfies $U_\mu U^\mu = -c^2$.

- (e) (Optional) For a particle with four momentum $P^\mu = (\frac{E}{c}, \mathbf{p}) = mU^\mu$ show that $P_\mu P^\mu = -(mc^2)^2/c^2$. This determines $E(\mathbf{p})$ the relation between energy and momentum:

$$\frac{E(\mathbf{p})}{c} = \sqrt{\mathbf{p}^2 + (mc)^2}. \quad (15)$$

- (i) Show the velocity of the particle (i.e. the group velocity) is

$$\mathbf{v}_p \equiv \frac{\partial E(\mathbf{p})}{\partial \mathbf{p}} = \frac{c^2 \mathbf{p}}{E} \quad (16)$$

- (f) (Do me! Not optional) A particle with velocity v_p in the x direction. Using the 4-vector transformation properties of U^μ , show that according to an observer moving to the right with velocity v , the particle moves with velocity

$$\underline{v}_p = \frac{v_p - v}{1 - v_p v / c^2} \quad (17)$$

Problem 6. One liners

- (a) Starting from the Maxwell equations for $F^{\mu\nu}$ and the definition of $F^{\mu\nu}$, derive the wave equation $-\square A^\mu = J^\mu/c$.
- (b) Starting from the maxwell equations for $F^{\mu\nu}$ in covariant form, show that we must have $\partial_\mu J^\mu = 0$ for consistency.
- (c) (This is two lines) Show that the energy conservation and force laws

$$\frac{dE_{\mathbf{p}}}{dt} = q\mathbf{E} \cdot \mathbf{v}_p \quad (18)$$

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}_p}{c} \times \mathbf{B}\right) \quad (19)$$

can be written covariantly

$$\frac{dP^\mu}{d\tau} = F^\mu{}_\nu u^\nu/c \quad (20)$$

Note that $E_{\mathbf{p}}$ (the energy of the particle) is different from \mathbf{E} the electric field.

- (d) From Eq. (20) show that $P_\mu P^\mu$ is constant in time.
- (e) Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under the gauge transform

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(X) \quad (21)$$

where Λ is an arbitrary function of $X = (t, \mathbf{r})$.

- (f) Given $F^{\mu\nu}$ the only two Lorentz invariant quantities are $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$. Evaluate these two invariants in terms of \mathbf{E} and \mathbf{B} ¹

¹ answers: $2(B^2 - E^2)$ and $-4\mathbf{E} \cdot \mathbf{B}$