Problem 1. Waves for the gauge potentials

(a) For the coulomb gauge, analyze the equations Maxwell equations for in vacuum

$$\varphi(t,\boldsymbol{x}) = \alpha\, e^{-i\omega t + i\boldsymbol{k}\boldsymbol{x}} \tag{1}$$

$$\boldsymbol{A}(t,\boldsymbol{x}) = \vec{\mathscr{A}}\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{2}$$

What are the conditions on α and $\vec{\mathscr{A}}$? What is the direction of $\vec{\mathscr{A}}$ relative to $\boldsymbol{k}$? What are the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in this gauge? Conclude that there are two degrees of freedom (in α and $\vec{\mathscr{A}}$) and that these two degrees of freedom determine $\boldsymbol{E}$ and $\boldsymbol{B}$.

(b) For the Lorenz gauge α and $\vec{\mathscr{A}}$ are different from the Coulomb gauge. What are the conditions on α and $\vec{\mathscr{A}}$ in this gauge. Hint: decompose $\vec{\mathscr{A}}$ into pieces longitudinal, $\vec{\mathscr{A}}_L$ and transverse $\vec{\mathscr{A}}_T$ to the propoagation direction $\boldsymbol{k}$.

$$\boldsymbol{k}\cdot\vec{\mathscr{A}}_T = 0 \tag{3}$$

$$\boldsymbol{k}\times\vec{\mathscr{A}}_L = 0 \tag{4}$$

Conclude that there are three degrees of freedom (in α and $\vec{\mathscr{A}}$) and but that only two of these degrees of freedom determine $\boldsymbol{E}$ and $\boldsymbol{B}$. The currents parallel to the line of sight determine $\vec{\mathscr{A}}_L$.

(c) In the Lorenz gauge, determine the electric and magnetic fields, and show that only $\vec{\mathscr{A}}_T$ contributes to the fields.

(d) Recall that under gauge transformation, we choose some $\Lambda(t,\boldsymbol{x})$ at will, and change the potentials according to this rule

$$\varphi(t,\boldsymbol{x}) \to \underline{\varphi}(t,\boldsymbol{x}) = \varphi(t,\boldsymbol{x}) - \partial_t\Lambda(t,\boldsymbol{x}) \tag{5}$$

$$A_i(t,\boldsymbol{x}) \to \underline{A}_i(t,\boldsymbol{x}) = A_i(t,\boldsymbol{x}) + \partial_i\Lambda(t,\boldsymbol{x}) \tag{6}$$

This does not change the $\boldsymbol{E},\boldsymbol{B}$ fields. Find the gauge transformation which converts the coulomb gauge waves into the Lorenz gauge.