6 Magneto Statics and Magnetic Matter

6.1 Magneto-Statics

At first order in $1/c$ we have the magneto static equations

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}_{tot}}{c} \qquad \boldsymbol{j}_{\rm tot} = \frac{\boldsymbol{j}}{c} + \underbrace{\frac{1}{c}\partial_t \boldsymbol{E}^{(0)}}_{\text{displacement current}} \tag{6.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.2}$$

where $\boldsymbol{j}_D = 1/c\,\partial_t \boldsymbol{E}^{(0)}$ is the displacement current. The formulas given below assume that $\boldsymbol{j}_D$ is zero. But, with no exceptions apply if one replaces $\boldsymbol{j} \to \boldsymbol{j} + \boldsymbol{j}_D$.

The current is taken to be steady

$$\nabla \cdot \boldsymbol{j} = 0 \tag{6.3}$$

Computing Fields: Lecture 14 and 15

(a) Below we note that for a current carrying wire

$$\boldsymbol{j}\,\mathrm{d}^3x = I\,\mathrm{d}\boldsymbol{\ell} \tag{6.4}$$

(b) We can compute the fields using the integral form of Ampères law $\nabla \times \boldsymbol{B} = j/c$, which says that the loop integral of $\boldsymbol{B}$ is equal to the current piercing the area bounded by the loop

$$\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{I_{\rm pierce}}{c} \tag{6.5}$$

For the familiar case of a current carrying wire we found $B_\phi = (I/c)/2\pi\rho$, where ρ is the distance from the wire.

(c) The Biot-Savat Law is seemingly similar to the coulomb law

$$\boldsymbol{B}(\boldsymbol{r}) = \int \mathrm{d}^3x_o \, \frac{\boldsymbol{j}(\boldsymbol{r}_o)/c \times \widehat{\boldsymbol{r} - \boldsymbol{r}_o}}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|^2} \tag{6.6}$$

We used this to compute the magnetic field of a ring of radius on the z-axis

$$B_z = 2\frac{(I/c)\pi a^2}{4\pi\sqrt{z^2 + a^2}} \tag{6.7}$$

which you can remember by knowing magnetic moment of the ring and other facts about magnetic dipoles (see below)

(d) Using the fact that $\nabla \cdot \boldsymbol{B} = 0$ we can write it as the curl of $\boldsymbol{A}$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad \boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\Lambda \tag{6.8}$$

but recognize that we can always add a gradient of a scalar function Λ to $\boldsymbol{A}$ without changing $\boldsymbol{B}$.

(e) If we adopt the coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$ and use the much used identity

$$\nabla \times (\nabla \times A) = -\nabla^2 \boldsymbol{A} + \boldsymbol{\nabla}(\nabla \cdot \boldsymbol{A}), \tag{6.9}$$

we get the result

$$-\nabla^2 \boldsymbol{A} = \frac{\boldsymbol{j}}{c}. \tag{6.10}$$

Then in free space $\boldsymbol{A}$ satisfies

$$\boldsymbol{A}(\boldsymbol{r}) = \int d^3x_o \frac{\boldsymbol{j}(\boldsymbol{r}_o)/c}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} \tag{6.11}$$

(f) The equations must be supplemented by boundary conditions. In vacuum we have that the parallel components of $\boldsymbol{B}$ jump according to size of the surface currents $\boldsymbol{K}$, while the normal components of $\boldsymbol{B}$ are continuous

$$\boldsymbol{n} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \frac{\boldsymbol{K}}{c} \tag{6.12}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{6.13}$$

Here $\boldsymbol{K}$ is the surface current and has units charge/length/s.

Multipole expansion of magnetic fields: Lecture 16

We wish to compute the magnetic field far from a localized set of currents. We can start with Eq. (6.14) and determine that far from the sources the vector potential is described by the magnetic dipole moment:

(a) The vector potential is

$$\boldsymbol{A} = \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{6.14}$$

where

$$\boldsymbol{m} = \tfrac{1}{2} \int d^3x_o \boldsymbol{r}_o \times \boldsymbol{j}(\boldsymbol{r}_o)/c \tag{6.15}$$

is the magnetic dipole moment.

(b) For a current carrying wire:

$$\boldsymbol{m} = \frac{I}{c}\frac{1}{2} \oint \boldsymbol{r}_o \times d\boldsymbol{\ell}_o = \frac{I}{c}\boldsymbol{a} \tag{6.16}$$

(c) The magnetic field from a dipole

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{3(\boldsymbol{n} \cdot \boldsymbol{m}) - \boldsymbol{m}}{4\pi r^3} \tag{6.17}$$

(d) **UNITS NOTE:** I defined $\boldsymbol{m}$ in Eq. (6.15) with $\boldsymbol{j}/c$. This has the "feature" that that

$$\boldsymbol{m}_{HL} = \frac{\boldsymbol{m}_{MKS}}{c} \tag{6.18}$$

In MKS units

$$\boldsymbol{A}_{MKS} = \mu_o \frac{\boldsymbol{m}_{MKS} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{6.19}$$

Setting $\varepsilon_o = 1$ so $\mu_o = 1/c^2$ and multiplying by c

$$\boldsymbol{A}_{HL} = c\boldsymbol{A}_{MKS} = \frac{\boldsymbol{m}_{MKS}/c \times \hat{\boldsymbol{r}}}{4\pi r^2} = \frac{\boldsymbol{m}_{HL} \times \hat{\boldsymbol{r}}}{4\pi r^2} \tag{6.20}$$

Below we will define the magnetization, and similarly $\boldsymbol{M}_{HL} = \boldsymbol{M}_{MKS}/c$.

Forces on currents

(a) We wish to compute the force on a small current carrying object in an external magnetic field. For a compact region of current (which is small compared to the inverse gradients of the external magnetic field) the total magnetic force is

$$\boldsymbol{F}(\boldsymbol{r}_o) = (\boldsymbol{m}\cdot\nabla)\,\boldsymbol{B}(\boldsymbol{r}_o) \tag{6.21}$$

where $\boldsymbol{m}$ is measured with respect $\boldsymbol{r}_o$, *i.e.*

$$\boldsymbol{m} = \frac{1}{2}\int_V \mathrm{d}^3x\;\delta\boldsymbol{r}\times\boldsymbol{j}(\boldsymbol{r})/c \tag{6.22}$$

with $\delta\boldsymbol{r} = \boldsymbol{r}-\boldsymbol{r}_o$.

(b) For a fixed dipole magnitude we have $\boldsymbol{F} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B})$ or

$$U(\boldsymbol{r}_o) = -\boldsymbol{m}\cdot\boldsymbol{B}(\boldsymbol{r}_o) \tag{6.23}$$

This formula is the same as the MKS one since we have taken $\boldsymbol{m}_{HL} = \boldsymbol{m}_{MKS}/c$.

(c) The torque is

$$\boldsymbol{\tau} = \boldsymbol{m}\times\boldsymbol{B} \tag{6.24}$$

(d) Finally (we will discuss this later) the magnetic force on a current carrying region is

$$(\boldsymbol{F}_B)^j = \frac{1}{c}\int_V (\boldsymbol{j}\times\boldsymbol{B})^j = -\int_{\partial V} dS\,\boldsymbol{n}_i T_B^{ij} \tag{6.25}$$

where

$$T_B^{ij} = -B^iB^j + \frac{1}{2}\boldsymbol{B}^2\delta^{ij} \tag{6.26}$$

is the magnetic stress tensor and $\boldsymbol{n}$ is an outward directed normal.

Solving for magneto-static fields

(a) One approach is to use direct integration:

$$\boldsymbol{A}(\boldsymbol{r}) = \mu\int \mathrm{d}^3x_o \frac{\boldsymbol{j}(\boldsymbol{r}_o)}{4\pi|\boldsymbol{r}-\boldsymbol{r}_o|}$$

Then for any current distribution once can compute the magnetic field – see lecture for an example of a rotating charged sphere . This is analogous to using the coulomb law.

(b) Another approach is to view

$$-\nabla^2\boldsymbol{A} = \mu\frac{\boldsymbol{j}}{c} \tag{6.27}$$

as a differential equation and to try separation of variables. There are (at least) two cases where the equations for $\boldsymbol{A}$ simplify.

i) If the current is azimuthally symmetric then it is reasonable to try a form $A_\phi(r,\theta)$

$$-\nabla^2\boldsymbol{A} = \mu\frac{\boldsymbol{j}}{c} \Rightarrow -\nabla^2A_\phi + \frac{A_\phi}{r^2\sin^2\theta} = \mu\frac{j_\phi}{c} \tag{6.28}$$

Here the $-\nabla^2A_\phi$ is the scalar Laplacian in spherical coordinates. For instance, this is an effective way to find the magnetic field from a ring of current or a rotating charged sphere.

ii) If the current runs up and down then you can try $A_z(\rho,\phi)$ in cylindrical coordinates:

$$-\nabla^2A_z(\rho,\phi) = \mu\frac{j_z}{c} \tag{6.29}$$

Here ∇^2A_z is the scalar Laplacian in cylindrical coordinates. See homework for an example of a cylindrical shell.

(c) Finally if the current separates two (or more) distinct regions of space (such as in a rotating charged sphere), then in each region one has

$$\nabla \times \boldsymbol{H} = 0 \tag{6.30}$$

So for each region one can introduce a scalar potential ψ_m such that

$$\boldsymbol{H} = -\nabla\psi_m \tag{6.31}$$

and (using $\nabla \cdot \boldsymbol{B} = 0$) show that

$$-\mu\nabla^2\psi_m = 0 \tag{6.32}$$

assuming μ is constant. Then the Laplace equation is solved in each region, and the boundary conditions (Eq. (6.49)) are used to connect the scalar potential across regions. The boundary conditions are markedly different from the electrostatic case, and this leads to markedly different solutions. See lecture for an example of the magnetic moment induced by an external field.

6.2 Magnetic Matter

Basic equations

(a) We are considering materials in the presence of a magnetic field. We write $\boldsymbol{j}_{\rm mat}$ (the medium (material) currents) as an expansion in terms of the derivatives in the magnetic field. For weak fields, and an isotropic medium , the lowest term in the derivative expansion, for a parity and time-reversal invariant material is

$$\frac{\boldsymbol{j}_{\rm mat}}{c} = \chi_m^B \nabla \times \boldsymbol{B} \tag{6.33}$$

where we have inserted a factor of c for later convenience.

(b) The current takes the form

$$\frac{\boldsymbol{j}_{\rm mat}}{c} = \nabla \times \boldsymbol{M} \tag{6.34}$$

i) $\boldsymbol{M}$ is known as the magnetization, and can be interpreted as the magnetic dipole moment per volume.

ii) We have worked with linear response for an isotropic medium where

$$\boldsymbol{M} = \chi_m^B \boldsymbol{B} \tag{6.35}$$

This is most often what we will assume.

iii) Usually people work with $\boldsymbol{H}$ (see the next items (c), (d) for the definition of $\boldsymbol{H}$) not $\boldsymbol{B}$ [1]

$$\boldsymbol{M} = \chi_m \boldsymbol{H} \tag{6.36}$$

iv) For not-that soft ferromagnets $\boldsymbol{M}(\boldsymbol{B})$ can be a very non-linear function of $\boldsymbol{B}$. This will need to be specified (usually by experiment) before any problems can be solved. Usually this is expressed as the magnetic field as a function of $\boldsymbol{H}$

$$\boldsymbol{B}(\boldsymbol{H}) \tag{6.37}$$

where $\boldsymbol{H}$ is small (of order gauss) and $\boldsymbol{B}$ is large (of order Tesla)

(c) After specifying the currents in matter, Maxwell equations take the form

$$\nabla \times \boldsymbol{B} = \nabla \times \boldsymbol{M} + \frac{\boldsymbol{j}_{ext}}{c} \tag{6.38}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.39}$$

[1] There are a couple of reasons for this. One reason is because the parallel components of H are continuous across the sample. But, ultimately it is $\boldsymbol{B}$ which is the curl $\boldsymbol{A}$, and it is ultimately the average current which responds to the gauge potential, through a retarded medium current-current correlation function that we wish to categorize.

or

$$\nabla \times \boldsymbol{H} = \frac{\boldsymbol{j}_{ext}}{c} \tag{6.40}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.41}$$

where [2]

$$\boldsymbol{H} = \boldsymbol{B} - \boldsymbol{M} \tag{6.43}$$

(d) For linear materials :

$$\boldsymbol{B} = \mu \boldsymbol{H} = \frac{1}{1 - \chi_m^B} \boldsymbol{H} = (1 + \chi_m) \boldsymbol{H} \tag{6.44}$$

Implying the definitions

$$\mu \equiv \frac{1}{1 - \chi_m^B} \equiv (1 + \chi_m) \tag{6.45}$$

Solving magnetostatic problems with linear magnetic media:

All of the methods described in Sect. (6.1) will work with minor modifications due to the boundary conditions described below

(a) For linear materials in the coulomb gauge we get

$$\nabla \times \boldsymbol{H} = \mu \frac{\boldsymbol{j}_{ext}}{c} \tag{6.46}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{6.47}$$

and with $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and constant μ we find

$$-\nabla^2 \boldsymbol{A} = \mu \frac{j_{ext}}{c} \tag{6.48}$$

which can be solved using the methods of magnetostatics.

(b) To solve magneto static equations we have boundary conditions:

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \frac{\boldsymbol{K}_{ext}}{c} \tag{6.49}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{6.50}$$

i.e. if there are no external currents then the parallel components of $\boldsymbol{H}$ are continuous and the perpendicular components of $\boldsymbol{B}$ are continuous.

(c) At an interface the there are bound currents which are generated

$$\boldsymbol{n} \times (\boldsymbol{M}_2 - \boldsymbol{M}_1) = \frac{\boldsymbol{K}_{mat}}{c} \tag{6.51}$$

[2] In the MKS system one has $\boldsymbol{H}_{MKS} = \frac{1}{\mu_o} \boldsymbol{B}_{MKS} - M_{MKS}$ so that $\boldsymbol{B}$ and $\boldsymbol{H}$ have different units. In a system of units where $\varepsilon_o = 1$ (so $1/\mu_o = c^2$) we have $H_{HL} = H_{MKS}/c$, $\boldsymbol{M}_{HL} = \boldsymbol{M}_{MKS}/c$ or since $1/c = \sqrt{\mu_o}$:

$$\boldsymbol{H}_{HL} = \sqrt{\mu_o} H_{MKS} \qquad \boldsymbol{M}_{HL} = \sqrt{\mu_o} \boldsymbol{M}_{MKS} \tag{6.42}$$