A Lagrangian for the Fields

$$I_0[A] = \int \text{all possible lorentz invariants consistent with symmetries and no more than quadratic}$$

Given field strength $F^{\mu\nu}$ there are two possible Lorentz invariant forms

- $F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$
- $F_{\mu\nu} \tilde{F}^{\mu\nu} = -4 \vec{E} \cdot \vec{B}$

This is parity odd. Lagrangian in nature appears to be parity even, $a_2 = 0$

So the general form is

$$I[A] = \int d^4x \; a_1 F_{\mu\nu} F^{\mu\nu} + \cancel{a_2 F_{\mu\nu} \tilde{F}^{\mu\nu}}$$

Choose this coefficient to be $-1/4$

- The factor $1/4$ is conventional.

- The (-1) in $-1/4$ is chosen so that we have, $L = KE - PE$.

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(E^2 - B^2)$$

like PE $\sim (\nabla \times A)^2$

$E^2 \sim (\partial_t \vec{A})^2 \sim$ like kinetic energy

Then

$$I_0 = \int d^4x \; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

write $A_\mu \to A_\mu + \delta A_\mu$ and expand see next pages

$$\delta I_0 = \int d^4x \; -\frac{1}{2} F^{\mu\nu} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)$$

by parts

$$= \int d^4x \; +\frac{1}{2} (\partial_\mu F^{\mu\nu} \delta A_\nu - \partial_\nu F^{\mu\nu} \delta A_\mu)$$

Relabel and use antisymm of $F^{\mu\nu}$

$$= \int d^4x \; \delta A_\beta \left[\partial_\alpha F^{\alpha\beta} \right]$$

In general the field is coupled to currents

$$I_{int} = \int d^4x \; J^\mu \frac{A_\mu}{c}$$

For example for the particle lagrangian

$$I_{int} = \int d\tau \; e \frac{dx^\mu}{d\tau} \frac{A_\mu}{c}$$

In general, define the current as

$$\delta I_{int} = \int d^4x \, J^{\mu} \, \delta A_{\mu}$$

or this is written,

$$\frac{\delta I_{int}}{\delta A_{\mu}(x)} = J^{\mu}(x)$$ but this, means this

Then

$$\delta I_0 + \delta I_{int} = \int d^4x \, \delta A_{\beta} \left[\partial_{\alpha} F^{\alpha\beta} + J^{\beta} \right]$$

Leading to the field eqs

$$\underbrace{-\partial_{\alpha} F^{\alpha\beta}} = \underbrace{J^{\beta}}$$

$$\underbrace{-\frac{\delta I_0}{\delta A_{\beta}}}_{\text{analagons to: } ma} = \underbrace{\frac{\delta I_{int}}{\delta A_{\beta}}}_{\text{analagons to force}}$$

Slow motion variation of F^2

$$\delta F^2 = \delta(F^{\mu\nu}F_{\mu\nu}) = \delta F^{\mu\nu}\,F_{\mu\nu} + F^{\mu\nu}\,\delta F_{\mu\nu}$$

$$= \delta F_{\mu\nu}\,F^{\mu\nu} + F^{\mu\nu}\,\delta F_{\mu\nu}$$

$$= 2\,F^{\mu\nu}\,\delta F_{\mu\nu}$$

Now

$$\delta F_{\mu\nu} = (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)$$

Gauge invariance & Current Conservation

Consider the interaction between the currents and the maxwell field

$$I_{int}[A]$$

Lets assume that this interaction is gauge invariant

current by definition

$$\delta I_{int} = \frac{1}{c}\int d^4x \, J^\mu \, \delta A_\mu$$

Now if I make a gaue transformation this does not change the value of $I_{int}[A]$ or $\delta I_{int}[A]$ Since the interaction is gauge invariant.

$$\delta A_\mu \longrightarrow \delta A_\mu + \frac{\partial \delta\Lambda}{\partial x^\mu} \qquad\qquad A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$$

Then

$$\delta I_{int} = \underbrace{\frac{1}{c}\int d^4x \, J^\mu \, \delta A_\mu}_{\delta I_{int}} + \frac{1}{c}\int d^4x \, J^\mu \frac{\partial \delta\Lambda}{\partial x^\mu}$$

So

$$0 = \frac{1}{c}\int d^4x \, J^\mu \frac{\partial \delta\Lambda}{\partial x^\mu}$$

Integrating by parts

$$0 = -\int d^4x \left(\frac{\partial J^\mu}{\partial x^\mu}\right)\delta\Lambda \Rightarrow \boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0}$$

In class problems

- Work in the confines of electrostatics. Show that the action takes the form

$$I = \int d^4x \ \frac{1}{2}(-\nabla\phi)^2 - \rho\phi$$

And that the variation of the action with respect to ϕ gives the Poisson Eq.:

$$-\nabla^2\phi = \rho$$

- The interaction lagrangian of a point particle is, where $x_0^\mu(\tau)$ is the trajectory of the particle

$$I_{int} = \frac{e}{c}\int d\tau \ \frac{dx_0^\mu}{d\tau} A_\mu(x_0(\tau))$$

Show that the current is

$$J^\mu = (c\rho, \vec{j}) = \left(ec\delta^3(\vec{x} - \vec{x}_0(t)), \ e\vec{v}\,\delta^3(\vec{x} - \vec{x}_0(t))\right)$$

Hint, start by writing

$$I_{int} = \int d^4x \int d\tau \ \frac{dx_0^\mu}{d\tau} \ A_\mu(x) \ \delta^4(x - x_0(\tau))$$

and then vary $A_\mu(x)$

Solution (1)

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(E^2 - \underbrace{\cancel{B^2}}_{0 \text{ for E-statics}}) = \frac{1}{2}E^2 = \frac{1}{2}(-\nabla\phi)^2$$

Similarly

$$\int d^4x \, \frac{J^\mu}{c} A_\mu = \int d^4x \left[\frac{(c\rho)}{c}(-\phi) + \underbrace{\cancel{\frac{\vec{j}}{c}}}_{0 \text{ for e-statics}} \cdot \vec{A} \right]$$

we used $J^\mu = (c\rho, \vec{j})$ $A^\mu = (\varphi, \vec{A})$ $A_\mu = (-\varphi, \vec{A})$

So

$$\int d^4x \, \frac{J^\mu}{c} A_\mu = -\int d^4x \; \rho\,\phi$$

Then

$$I_{tot} = \int d^4x \; \frac{1}{2}(\nabla\phi)^2 - \rho\phi$$

$$\delta I = \int d^4x \left[\nabla\phi \, \nabla\delta\phi - \rho\,\delta\phi \right]$$

$$= \int d^4x \; \delta\phi \left[-\nabla^2\phi - \rho \right]$$

Or

$$-\nabla^2\phi = \rho$$

Solution (2)

$$\delta I_{int} = \frac{e}{c}\int d^4x \left[\int d\tau \, \frac{dx_0^\mu}{d\tau} \, \delta^4(x - x_0(\tau))\right] \delta A_\mu(x)$$

So

$$\frac{J^\mu}{c} = \frac{e}{c}\int d\tau \, \frac{dx_0^\mu}{d\tau} \, \delta^4(x - x_0(\tau))$$

Then we integrate over τ, $d\tau = \frac{dt}{\gamma}$, with $\frac{dx_0^\mu}{d\tau} = (\gamma c, \gamma \vec{v})$.

$$J^\mu = e \int dt \, (c, \vec{v}) \, \delta^4(x - x_0(t))$$

$$\boxed{J^\mu = \left(ec\, \delta^3(\vec{x} - \vec{x}_0(t)),\ e\vec{v}\, \delta^3(\vec{x} - \vec{x}_0(t))\right)}$$