Problem 1. Basics of Relativity

(a) The space time event at $X^\mu = (X^0, X^i) = (ct, \boldsymbol{x})$ happens at $\underline{X}^\mu = (\underline{X}^0, \underline{X}^i) = (c\underline{t}, \underline{\boldsymbol{x}})$ according to an observer moving to the right along the x axis with velocity v. Define the "light-cone" coordinates $x^+ \equiv X^0 + X^1$ and $x^- \equiv X^0 - X^1$. Show that under this boost that the x^+ coordinates are contracted, while the x^- coordinates are elongated

$$\underline{x}^+ = e^{-y} x^+ = \sqrt{\frac{1-\beta}{1+\beta}} x^+ \, , \tag{1}$$

$$\underline{x}^- = e^{y} x^- = \sqrt{\frac{1+\beta}{1-\beta}} x^- \, . \tag{2}$$

Here

$$y = \tanh^{-1} \beta = \frac{1}{2} \log \left(\frac{1+\beta}{1-\beta} \right) \tag{3}$$

is the so-called "rapidity" of the boost. What is $\underline{x}^+ \underline{x}^-$ and why is it unchanged under boost?

(b) A Lorentz tensor transforms as

$$\underline{T}^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma T^{\rho\sigma} \tag{4}$$

Show that the transformation rule can be alternatively written

$$\underline{T}^\mu{}_\nu = L^\mu{}_\rho T^\rho{}_\sigma (L^{-1})^\sigma{}_\nu \tag{5}$$

or equivalently

$$\underline{T}^\mu{}_\nu = L^\mu{}_\rho L_\nu{}^\sigma T^\rho{}_\sigma \tag{6}$$

(c) The frequency and wave number of a plane wave of light, $e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} = e^{iK\cdot X}$, form a lightlike four vector

$$K^\mu = \left(\frac{\omega}{c}, \boldsymbol{k} \right) \tag{7}$$

(i) Show that $K \cdot K = K_\mu K^\mu = 0$ (this is the statement that K is lightlike.)

(ii) If a photon has frequency ω_o and is propagating along the z-axis, show (using the 4-vector properties of K^μ) that according to an observer propagating in the negative z direction with speeed β

$$\omega = \sqrt{\frac{1+\beta}{1-\beta}} \omega_o \tag{8}$$

(d) Show that the four velocity $U^\mu = dx^\mu / d\tau$ satisfies $U_\mu U^\mu = -c^2$.

(e) For a particle with four momentum $P^\mu = (\frac{E}{c}, \boldsymbol{p}) = mU^\mu$ show that $P_\mu P^\mu = -(mc^2)^2/c^2$. This determines $E(\boldsymbol{p})$ the relation between energy and momentum:

$$\frac{E(\boldsymbol{p})}{c} = \sqrt{\boldsymbol{p}^2 + (mc)^2} \, . \tag{9}$$

(i) Show the velocity of the particle (i.e. the group velocity) is

$$\boldsymbol{v_p} \equiv \frac{\partial E(\boldsymbol{p})}{\partial \boldsymbol{p}} = \frac{c^2 \boldsymbol{p}}{E} \tag{10}$$

(f) A particle with velocity v_p in the x direction. Using the 4-vector transformation properties of U^μ, show that according to an observer moving to the right with velocity v, the particle moves with velocity

$$\underline{v}_p = \frac{v_p - v}{1 - v_p v/c^2} \tag{11}$$