Maxwell Equations and Units

We will rewrite the Maxwell equations in MKS units in Heavyside Lorentz units. This makes the role of the speed of light explicit.

In MKS:

$$\nabla\cdot E_{MKS} = \underset{\text{charge/vol}}{\rho}/\varepsilon_0$$

current/area = charge / (area·s)

$$\nabla\times B_{MKS} = \mu_0 \, \dot{j}_{MKS} + \mu_0\varepsilon_0 \frac{\partial E_{MKS}}{\partial t}$$

$$\nabla\cdot B_{MKS} = 0$$

$$\nabla\times E_{MKS} = -\frac{\partial B_{MKS}}{\partial t}$$

Then, defining $Q_{HL} = \dfrac{Q_{MKS}}{\sqrt{\varepsilon_0}}$, $E_{HL} = \sqrt{\varepsilon_0}\, E_{MKS}$, and

$B_{HL} = \dfrac{B_{MKS}}{\sqrt{\mu_0}}$ and $\dfrac{j_{HL}}{c} = \sqrt{\mu_0}\, j_{MKS}$ (since $j_{HL} = \dfrac{j_{MKS}}{\sqrt{\varepsilon_0}}$),

with $c = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}$, we find:

(This is motivated by:

$$dE_{MKS} = \frac{dQ_{MKS}}{4\pi\varepsilon_0 r^2} \quad \text{and} \quad dB_{MKS} = \frac{\mu_0}{4\pi}\frac{\vec{I}\times dl}{r^2}$$)

We find

$$\nabla \cdot E_{HL} = \rho$$

$$\nabla \times B_{HL} = \frac{j_{HL}}{c} + \frac{1}{c}\frac{\partial E_{HL}}{\partial t}$$

$$\nabla \cdot B_{HL} = 0$$

$$\nabla \times E_{HL} = -\frac{1}{c}\frac{\partial B_{HL}}{\partial t}$$

Note:

$$dE_{HL} = \frac{dQ_{HL}}{4\pi r^2}$$

$$dB_{HL} = \frac{I/c \, dl}{4\pi r^2}$$

Remarks

To understand this redefinition, first set $\varepsilon_0 = 1$ (i.e. use a specific units for charge). Then the maxwell equations read with $\mu_0 = \frac{1}{c^2}$ (since $\varepsilon_0 = 1$)

$$\nabla \cdot \bar{E} = \rho$$

$$\nabla \times \bar{B} = \frac{j}{c^2} + \frac{1}{c^2}\frac{\partial \bar{E}}{\partial t}$$

(mks with $\varepsilon_0 = 1$)

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

Formally we have defined, $\bar{E} = \sqrt{\varepsilon_0}\, E_{mks}$, $\bar{B} = \sqrt{\varepsilon_0}\, B_{mks}$, and $\bar{Q} = Q/\sqrt{\varepsilon_0}$ (so $\bar{\rho} = \rho/\sqrt{\varepsilon_0}$, $\bar{j} = j/\sqrt{\varepsilon_0}$)

Then, as we will see, E and B are very much the same thing. Thus we define

$$E_{HL} = \bar{E} \qquad B_{HL} = c\bar{B} \quad \text{so that}$$

E_{HL} and B_{HL} have the same units:

$$\nabla \cdot E_{HL} = \rho_{HL}$$

$$\nabla \times B_{HL} = \frac{j_{HL}}{c} + \frac{1}{c}\frac{\partial E_{HL}}{\partial t}$$

$$\nabla \cdot B_{HL} = 0$$

$$\nabla \times E_{HL} = -\frac{1}{c}\frac{\partial B_{HL}}{\partial t}$$

<u>Examples:</u>

To convert from MKS to HL, set $\varepsilon_0 = 1$ and $\mu_0 = 1/c^2$. Arrange to multiply B-fields by c.

$$F = q_{MKS}(E_{MKS} + V \times B_{MKS})$$

$$F = q_{MKS}\left(E_{MKS} + \left(\frac{V}{c}\right) \times (cB_{MKS})\right)$$

$$= q\left(E_{HL} + \frac{V}{c} \times B_{HL}\right)$$

MKS HL

$$dE = \frac{dQ}{4\pi\varepsilon_0 r^2} \longrightarrow dE_r = \frac{dQ}{4\pi r^2}$$

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2} \longrightarrow dB = \frac{I/c\,dl}{4\pi r^2}$$

Here, we set $\mu_0 = 1/c^2$ (with $\varepsilon_0 = 1$) and then $B_{HL} = cB_{mks}$ (with $\varepsilon_0 = 1$). Or one can just remember $B_{HL} = \frac{B_{mks}}{\sqrt{\mu_0}}$ and $\frac{I}{c}_{HL} = \sqrt{\mu_0}\, I_{mks}$.