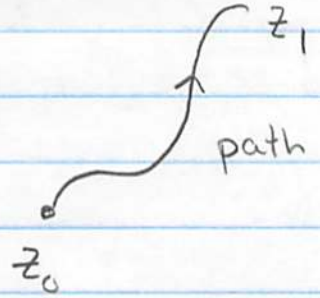
Integration



path in complex plane $\quad (x(t), y(t)) \equiv \gamma(t)$

real parameter of path

$$dx \equiv \dot{x}\, dt \qquad dy = \dot{y}\, dt$$

$$\int_{z_0}^{z_1} f\, dz = \int_{z_0}^{z_1} (u + iV)(dx + i dy)$$

$$= \int_{\text{path}} u\, dx - v\, dy \;+ i \int_{\text{path}} v\, dx + u\, dy$$

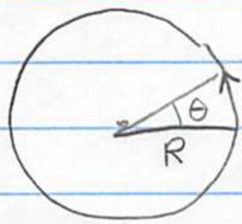
Example 1: This is important!

$$I = \oint_{\text{circle}} \frac{dz}{z}$$

Parametrize the path by θ:

$$z = Re^{i\theta} = R\cos\theta + iR\sin\theta$$

$$dz = Re^{i\theta}\, i d\theta = z\, i d\theta$$



Thus:

$$\oint_{\text{circle}} \frac{dz}{z} = \int_0^{2\pi} i\, d\theta = 2\pi i$$

no matter the size

Example 2 • Circle again

• Go twice around the circle the "clockwise"/ "wrong"/ "opposite" way $z = Re^{-i\theta}$

$$\oint \frac{dz}{z} = \int_0^{2(2\pi)} -i\,d\theta = (2\pi i)(-2)$$

twice around the wrong way

• Changing the direction changes the sign!!

Example 3 for n integer (positive or negative but not zero)

• $I = \oint \frac{dz}{z} z^n$

circle again

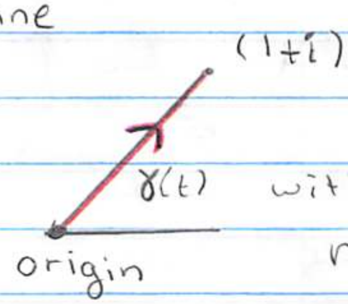
$$I = \int_0^{2\pi} i\,d\theta\; R^n e^{in\theta} = iR^n \int_0^{2\pi} e^{in\theta}\,d\theta$$

n integer

$I = 0$ idependent of R!

Example 4 • Straight line

$$I = \int_0^{z_1} z^n\,dz$$



with $t \in [0,1]$ a real number

$z(t) = t(1+i)$ $\quad dz = dt\,(1+i)$

$= t\,z_1$ $\quad dz = dt\,z_1$

Then

$$I = \int_0^1 (z_1 t)^n \, z_1 \, dt$$

$$= z_1^n \int_0^1 t^n \, dt$$

$$I = \frac{z_1^n}{n+1}$$ as you could have hoped!

For a general straight line connecting z_0 and z_1

$$I = \int_{z_0}^{z_1} dz \, (z - z_0)^n = \frac{(z_1 - z_0)^{n+1}}{n+1}$$

In fact the path, doesn't matter for holomorphic functions as we show next!

__Example 5__ Just shift example 1 and 3

$$\oint_{\text{Circle}} \frac{dz}{z - z_0} = 2\pi i \qquad \oint_{\text{Circle}} \frac{dz}{z - z_0} (z - z_0)^n = 0$$

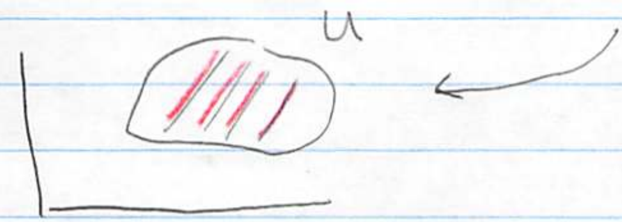
for n positive, negative, but not zero integer

Circle around z_0

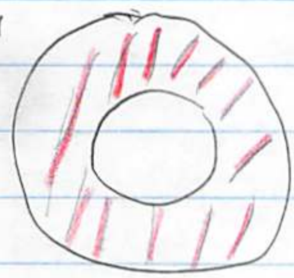
Prove it yourself. Set $(z - z_0) = Re^{i\theta}$ ← parameter of integration

Some Terminology

- An open simply connected set in th complex plane looks like this

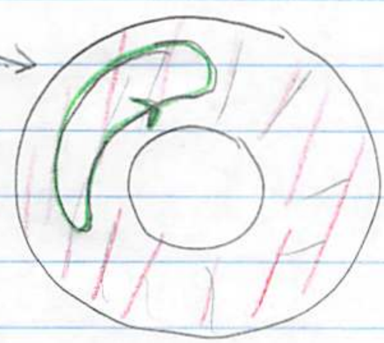


This is connected but not simply connected, because it has a hole in it

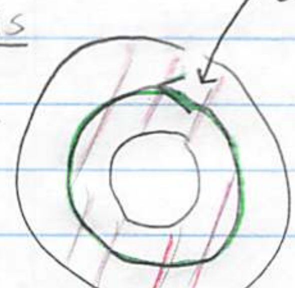


- A non-intesecting curve deformable to a point, or homologous / homotopic to a point, looks like:

this



not this



can't squeeze it to a point

- Every closed curve in a simply connected set is deformable to a point.

Cauchy Theorm in Homotopic Form

Consider open set in the complex plane. Assume that $f(x,y)$ is holomorphic in U

Then on any closed non-intersecting path, deformable to a point in U:

$$I = \oint_\gamma dz\, f(z) = 0$$

Proof

$$I = \oint_{path} u\,dx - v\,dy + i\oint_{path} v\,dx + u\,dy$$

In general the Stokes theorem says for $P(x,y)$, $Q(x,y)$

$$\oint_{loop} P\,dx + Q\,dy = \int_{area} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Yielding in this case

By cauchy Riemann

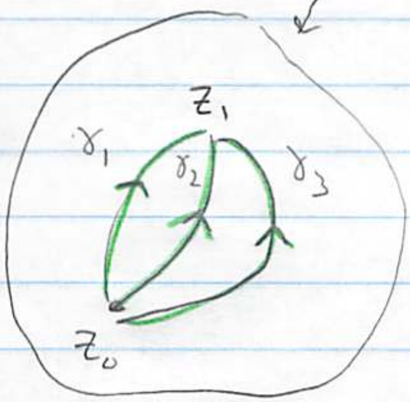
So

$$I = \int dx\,dy \cancelto{0}{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)} + i \int dx\,dx \cancelto{0}{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}$$

Examples

Open set U where $f(z)$ is holomorphic

(1) $\gamma_1, \gamma_2, \gamma_3$ are homotopic (homolgous) since you can deform γ_1 to γ_2 while remaining in U. The all give



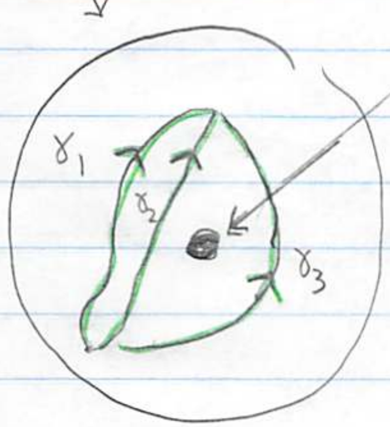
$$\int_{\gamma_1} = \int_{\gamma_2} = \int_{\gamma_3}$$

the same integral

$$\int_{z_0}^{z_1} f(z)\,dz$$

not simply connected, but still connected & open.

(2)



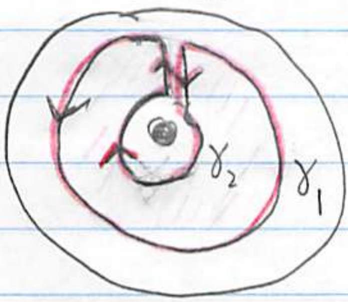
open set now excludes the origin (where for example the function is not holomorphic)

- γ_1 and γ_2 have the same integral. they are homotopic / homologous

- γ_1 and γ_3 are not homologous and their integrals are not the same

③

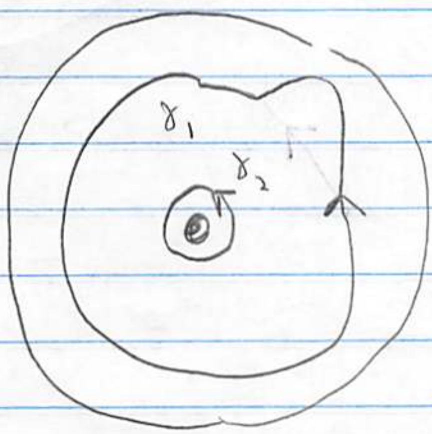
open set U



- The curve $\gamma_1 + \gamma_2$ is deformable to a point, gives zero for analytic functions

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = 0 \quad \text{i.e. } I_{\gamma_1} = -I_{\gamma_2}$$

④

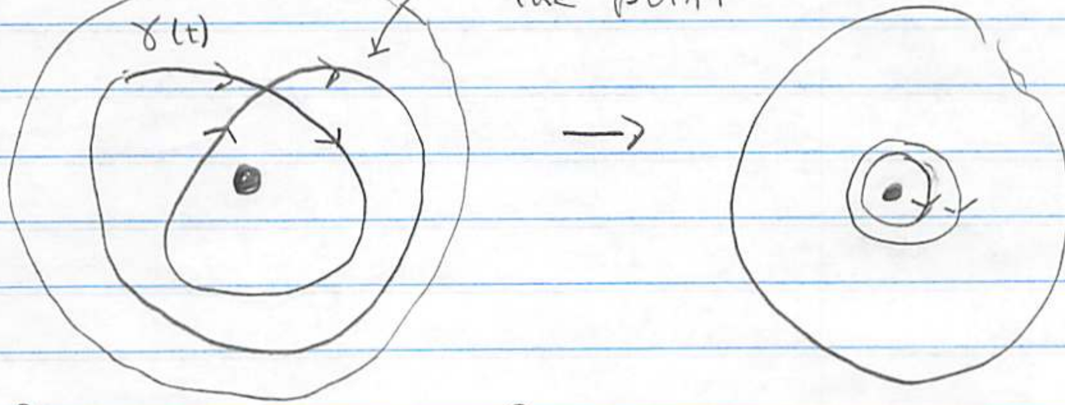


$$\oint_{\gamma_1} dz\, f(z) = \oint_{\gamma_2} dz\, f(z)$$

- The two cotours now point in the same direction (unlike Ex. 3)

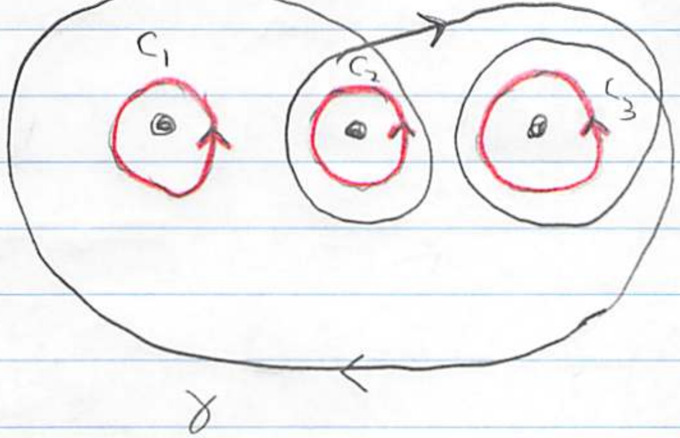
⑤

the curve winds twice around the point



$$\oint_{\gamma} dz\, f(z) = 2 \oint_{\text{circle}} dz\, f(z)$$

(6) One more:

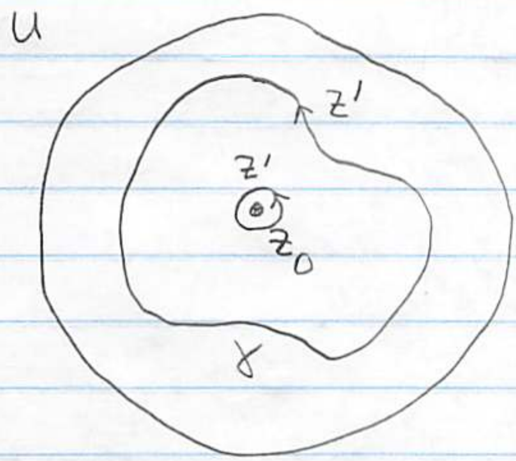


$$\oint_\gamma dz\, f(z) = -\oint_{C_1} f(z)\,dz - 2\oint_{C_2} f(z)\,dz - 2\oint_{C_3} f(z)\,dz$$

circle twice around, but γ goes clockwise while C_1, C_2, C_3 go counter-clockwise the "normal" way

Cauchy Formula, Holomorphy = analyticity

Then consider the integral of a holomorphic function around a path deformable to a point which contains z_0:



$$I = \oint_\gamma \frac{dz'}{z'-z_0} f(z')$$

Now deform γ to a small circle around z_0. We can do this because $f(z')$ and $1/(z'-z_0)$ are holomorphic except at z_0

$$I = \oint_{\text{Circle}} dz' \, \frac{f(z')}{z'-z_0}$$

But $f(z)$ is holomorphic / continuous at z_0 and is thus approximately constant $f(z') \simeq f(z_0)$ + tiny

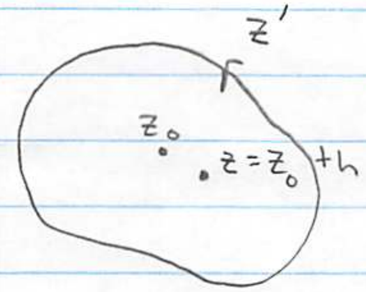
$$I = f(z_0) \oint \frac{dz'}{z'-z_0} = f(z_0)\, 2\pi i$$

And we find

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_\gamma dz' \, \frac{f(z')}{(z'-z_0)}}$$

There is more. Pick a point $z = z_0 + h$ close to z_0 with h small

$$f(z_0+h) = \frac{1}{2\pi i}\oint \frac{f(z')}{z'-(z_0+h)}$$



And expand in h:

$$\frac{1}{z'-(z_0+h)} = \frac{1}{z'-z_0}\;\frac{1}{\left(1-\frac{h}{z'-z_0}\right)}$$

this converges for h small

$$= \frac{1}{z'-z_0}\left[1 + \left(\frac{h}{z'-z_0}\right) + \left(\frac{h}{z'-z_0}\right)^2 + \dots\right]$$

Then

$$f(z_0+h) = \frac{1}{2\pi i}\oint f(z')\left[\frac{1}{z'-z_0} + \frac{h}{(z'-z_0)^2} + \frac{h^2}{(z'-z_0)^3} + \dots\right]$$

i.e.

$$\boxed{f(z_0+h) = \sum_{n=0}^{\infty} a_n h^n} \Leftarrow \text{taylor series}$$

with

$$\boxed{a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint dz' \frac{f(z')}{(z'-z_0)^{n+1}}}$$

- Thus we have shown that holomorphic functions (i.e. satisfying cauchy-riemann equations) are analytic by explicit construction.