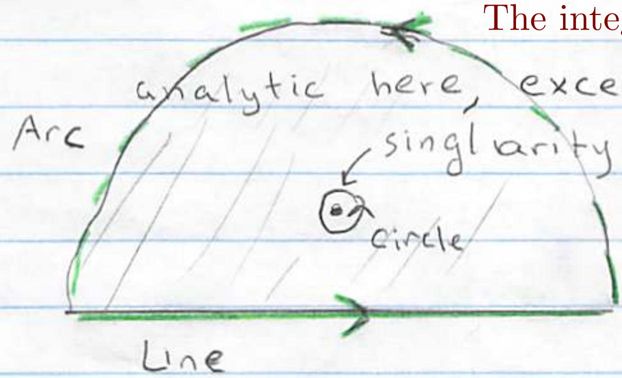
Example 1 Simple Poles

we will Show this!

- $$I = \int_{-\infty}^{\infty} dx \, \frac{1}{(x^2+1)} = \int_{\text{Line}} dz \, \frac{1}{(z+i)(z-i)} = \pi$$

$$(x^2+1) = (z+i)(z-i)$$

The integrand $1/(z^2+1)$ is analytic here except at $z = \pm i$



analytic here, except for the singularity at $z=i$

- Then add the arc at infinity which gives nothing:

$$I_{arc} = \int_{arc}^{\infty} dz \, \frac{1}{z^2+1} \leq \begin{pmatrix}\text{length}\\ \text{of arc}\end{pmatrix} \begin{pmatrix}\text{max of func}\\ \text{on arc}\end{pmatrix}$$

$$< \frac{2\pi R}{R^2} \xrightarrow[R\to\infty]{} 0$$

- Now deform the contor to a small circle around $z=i$:

$$I_{line} + \cancel{I_{Arc}}^{\,0} = I_{\substack{\text{small}\\ \text{Circle}}} \text{ at } i$$

we did this integral earlier

$$I_{\substack{\text{small}\\ \text{Circle}\\ \text{at } z=i}} = \oint_{\text{Circle}} \frac{1}{(z+i)(z-i)} = \frac{1}{2i} \oint dz \, \frac{1}{z-i} = \frac{1}{2i} \, 2\pi i = \pi$$

z is almost i on circle boundary $z+i = 2i$

Residues:

- This procedure gets formalized. If f has expansion at z_0 of the form:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

note negative powers here!

Then the residue is the coefficient of $\frac{1}{z-z_0}$.

- Thus, the residue of our function in Example ① is

$$\operatorname*{Res}_{z=i} \frac{1}{(z-i)(z+i)} = \frac{1}{2i}$$

$$2i = z+i \Big|_{z=i}$$

And we have a formula:

$$\boxed{\oint_\gamma f(z)\, dz = 2\pi i \sum_{z_a} \mathrm{Res}_{z_a} f}$$

- Here $f(z)$ is analytic, except for single poles at some set of points $\{z_a\}$. The integration is over a closed contour deformable to a point enclosing the $\{z_a\}$.

- In this case there are poles at $z = +i, -i$. We closed the contour above, encircling the upper pole. The residue there is $(2i)^{-1}$ and thus

$$I = 2\pi i \, \frac{1}{2i} = \pi$$

A note on terminology

Suppose $f(z)$ has an expansion at z_0 of the form

$$f(z) = \underbrace{\frac{a_{-2}}{(z-z_0)^2}}_{\text{double pole}} + \underbrace{\frac{a_{-1}}{(z-z_0)}}_{\text{single pole}} + a_0 + a_1(z-z_0) + \ldots$$

We say $f(z)$ has a double pole (or a pole of order two). The residue of $f(z)$ at z_0 is the coefficient of the single pole, i.e.

$$\operatorname*{Res}_{z=z_0} f = a_{-1}$$

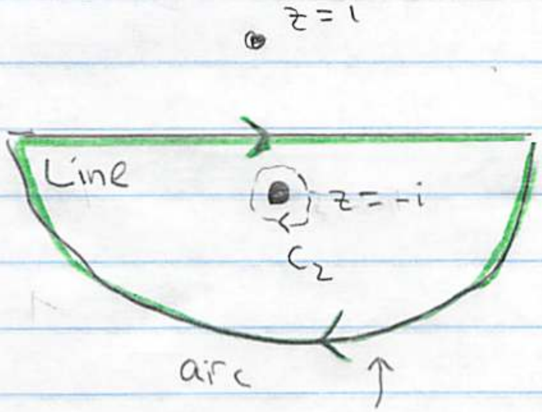
Example 2: Double Poles and Fourier Integrals

Consider

- $$I(k) = \int_{-\infty}^{\infty} dx \, \frac{e^{+ikx}}{(x^2+1)^2}$$ watch the two!

take $k<0$ for example ← watch me!

Then $(x^2+1)^2 = (z+i)^2(z-i)^2$. So



"clockwise = wrong way around"

$$I_{\text{line}} = \int_L dz \, \frac{e^{ikz}}{(z+i)^2(z-i)^2}$$

- Now since $k<0$, we must close the contour below. Take $z = iy$, then

$e^{ik(iy)} = e^{-ky}$. So if $y<0$ then $e^{-ky} \to 0$ (since $-k>0$), while if $y>0$, e^{-ky} increases and the arc at infinty diverges. So we close below, where $y<0$:

- $$I_{\text{line}} = \oint_{\substack{\text{arc} \\ +\text{line}}} dz \, \frac{e^{ikz}}{(z+i)^2(z-i)^2}$$

Uses Cauchy Theorem

$$= \oint_{C_2} dz \, \frac{e^{ikz}}{(z+i)^2(z-i)^2}$$

double pole

- Then the function near $z=-i$ takes the form

$$\frac{e^{ikz}}{(z+i)^2(z-i)^2} = \underbrace{\frac{a_{-2}}{(z+i)^2}}_{\text{double pole}} + \underbrace{\frac{a_{-1}}{(z+i)}}_{\text{single pole}} + a_0 + a_1(z+i) \ldots$$

We only care about the residue, a_{-1}, since

$$\oint \frac{1}{(z+i)^n} = 0 \quad \text{unless } n=-1$$

small
circle around
$z=-i$

- Expanding near $z=-i$ the function <u>in front of</u> $1/(z+i)^2$

$$\frac{e^{ikz}}{(z-i)^2} = g(z) \qquad g(z) = g(-i) + g'(-i)(z+i) + \frac{g''(-i)}{2!}(z+i)^2 + \ldots$$

Near $z=-i$

$$g(z) \simeq \frac{e^k}{(2i)^2} - ie^k\frac{(k-1)}{4}(z+i) + \ldots$$

see next page

- Then:

$$\frac{e^{ikz}}{(z+i)^2(z-i)^2} = \frac{e^k}{(2i)^2(z+i)^2} - \frac{ie^k(k-1)}{4}\frac{1}{(z+i)} + a_0 + \ldots$$

So finally

$$I_{line} = -2\pi i \operatorname{Res}_{z=-i} f$$

Because we circle the pole in a clockwise rather than counter-clockwise fashion:

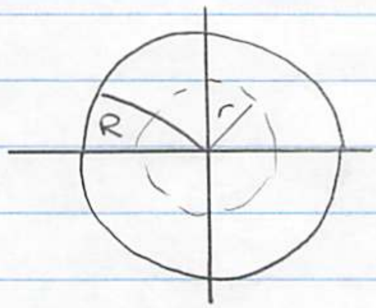
$$I_{line} = -2\pi i \left(\frac{-ie^k}{4} (k-1) \right)$$

$$I_{line} = \pi e^k \frac{(1-k)}{2}$$

Analytic Functions

- Take a power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$



The radius of convergence is the largest R such that for all $r < R$ the series:

$$\sum_n |a_n| r^n$$

converges. This is known as absolute uniform convergence. In general, the n-th term (n-large)

$$|a_n| r^n < 1$$

had better be less than unity or the successive terms in the expansion woulld get bigger and bigger. The boundary of convergence is thus

$$\lim_{n\to\infty} |a_n| R^n = 1 \qquad \text{i.e} \qquad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$$

I will not prove it but there is a useful formula:

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \quad \text{if the limit exists}$$

this the Cauchy Ratio Test

Example e^z converges in the entire complex plane

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots$$

Then we have $R = \infty$ or $1/R = 0$ from these steps:

$$\frac{1}{R} = \lim_{n \to \infty} \frac{1}{(n!)^{1/n}} = \frac{1}{(n^n e^{-n})^{1/n}} = \lim_{n \to \infty} \left(\frac{e}{n} \right) = 0 .$$

We used the Stirling approximation for $n \to \infty$

$$n! \simeq n^n e^{-n} \qquad \text{or} \qquad \log n! = n \log n - n$$ try to prove me

More generally the next correction to the stirling approx is used $n! \simeq n^n e^{-n} \sqrt{2\pi n}$ and is quite accurate even for modest n.

- Proof of Stirling Approx:

$$\log n! = \log(1) + \log(2) + \log(3) + \ldots \log(n)$$

$$= \sum_{k=1}^{n} \log(k) \simeq \int_1^n dx \, \log(x)$$

$$\simeq x \log(x) - x \Big|_1^n$$

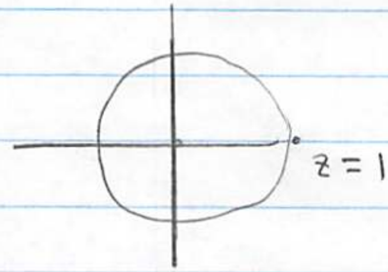
$$\simeq n \log(n) - n + \text{order unity}$$

So $n! \simeq n^n e^{-n}$

The Radius of Convergence and Singularites

- Let's look at

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots$$

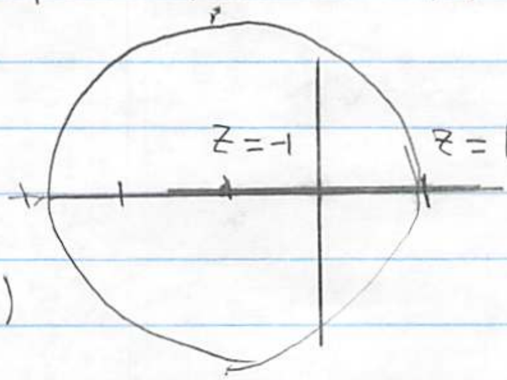


We showed that radius of convergence is $R = 1$, i.e. the distance between the origin and the nearest singularity.

- Similarly, take $1/1-z$ and expand near $z = -1$:

$$\frac{1}{1-z} = \frac{1}{2 - z + 1} = \frac{1}{2\left(1 - \frac{z+1}{2}\right)}$$

$$= \frac{1}{2}\left(1 + \left(\frac{z+1}{2}\right) + \left(\frac{z+1}{2}\right)^2 + \ldots\right)$$



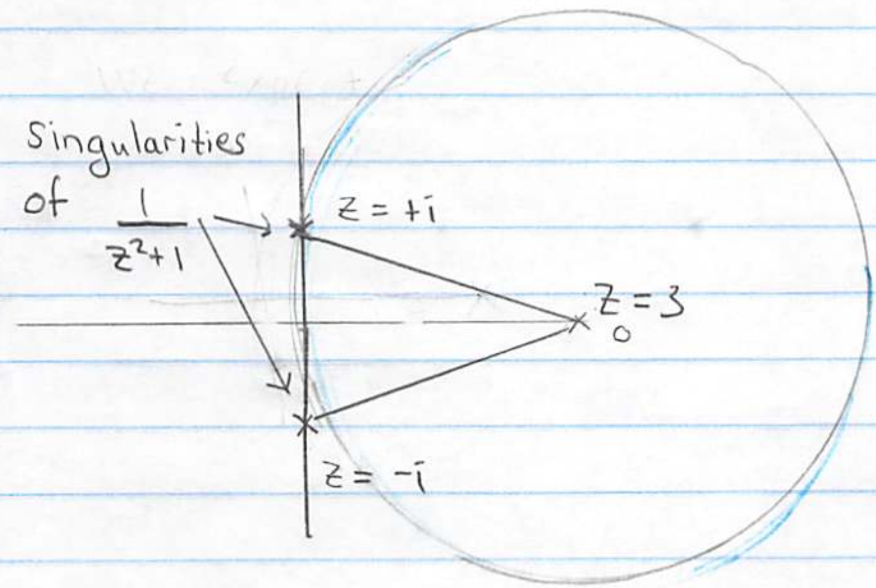
These terms are getting successively smaller provided $|z+1| < 2$. Here 2 is the distance between the expansion point and the nearest singularity in the complex plane

- This is the general result: The radius of convergence is the distance between the expansion point and the nearest singularity in the complex plane.

Thus, take for example,

$\frac{1}{z^2+1}$, and ask about its convergence at $z = 3$. The series is

$$\frac{1}{z^2+1} \simeq \frac{1}{10} - \frac{3}{10}(z-3) + \frac{13}{500}(z-3)^2 + \ldots$$

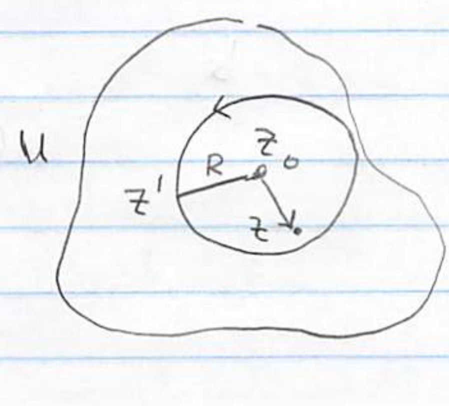


The distance between the expansion point $(z_0 = 3)$ and the nearest singularity $(z = \pm i)$ is $R = \sqrt{3^2+1^2} = \sqrt{10}$

This is the radius of Convergence

Proof Let U be the region where the fcn is analytic.

- Let R be largest disk in U that can be drawn around z_0. From the cauchy theorem



$$f(z) = \sum_n a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z')}{(z'-z_0)^{n+1}}$$

- Parametrize the circle by $z' - z_0 = Re^{i\theta}$ $dz' = iR\,d\theta$

Then, $f(\theta) = f(Re^{i\theta})$, and we find that a_n are:

$$a_n = \frac{1}{2\pi R^n} \int_0^{2\pi} d\theta\, e^{-in\theta} f(\theta)$$

We see that the a_n are the fourier coefficients of $f(\theta)$

$$\leq \frac{\|f\|}{R^n}$$

$\|f\| \equiv$ maximum value of $|f|$ on the circle boundary

Take a_0. a_0 is the average of $f(\theta)$. This is clearly less than the maximum of $f(\theta)$

- Thus the n-th term in the series scales as

on previous page
See picture

$$|a_n (z - z_0)^n| < \|f\| \left(\frac{|z - z_0|}{R}\right)^n \quad \text{with } \frac{|z-z_0|}{R} < 1$$

which clearly gets smaller and smaller geometrically so that the series converges (compare to $1/(1-z)$). So the radius of convergence is at least R.

But if it were more than R, the function would be analytic outside of R, contradicting the assumption that R is the maximum circle contained in the domain of analyticity U.