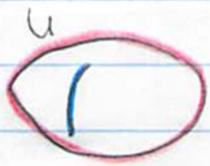
Analytic Continuation

① If an analytic function is zero on any line segment, then it is zero everywhere in its domain of definition U:

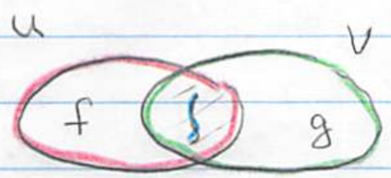


- Think about it. An analytic function is determined by a countable number of series coefficients $\{a_n\}$, while a line segment is a non-countable set of points.

$$f(z) = \sum_n a_n z^n$$

So if a function is zero on a line segment, then all of its derivates are zero there. There are more zero-conditions than coefficients $\{a_n\}$, leading to $a_n = 0$

② If two analytic functions agree on a line segment then they must agree everywhere in their common domain of definition $U \cap V$. Just look at $f - g$ on $U \cap V$ (with f defined on U and g defined on V) and return to ①



③ The function g defined on V (which agrees with f on a segment in $U \cap V$) is unique throughout V. To prove, just look at $g_1 - g_2$ defined on V. It is zero on a segment. So $g_1 - g_2 = 0$ throughout V by ①.

g is known as the (unique!) analytic continuation of f to the set V.

Ex 1

$$\frac{1}{z} = \frac{1}{(z-1)+1} \Rightarrow f_1(z) \equiv 1-(z-1)+(z-1)^2+\ldots$$

converges for $|z-1|<1$

- $f_1(z)$ has domain of definition $|z-1|<1$.

- Suppose I had another function analytic at $z=i$

$$f_2(z) = \sum_n a_n (z-i)^n$$

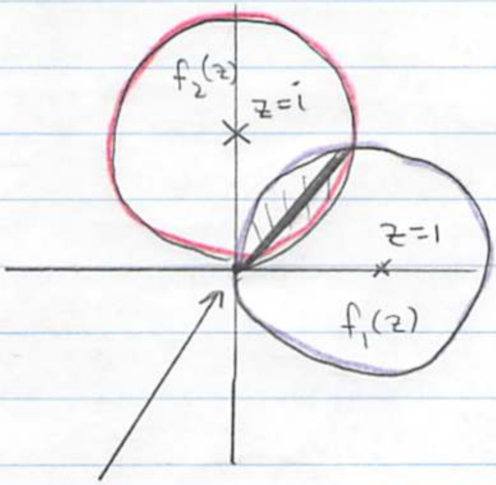
Then the only way $f_2(z)$ will agree with $f_1(z)$ is if the coefficients a_n are chosen to reproduce $1/z$

$$\frac{1}{z-i+i} = \frac{1}{z} \Rightarrow f_2(z) = \frac{1}{i} - \frac{(z-i)}{i^2} + \frac{(z-i)}{i^3} + \ldots$$

- Then $f_2(z)$ is defined for $|z-i|<1$.

The picture is the following

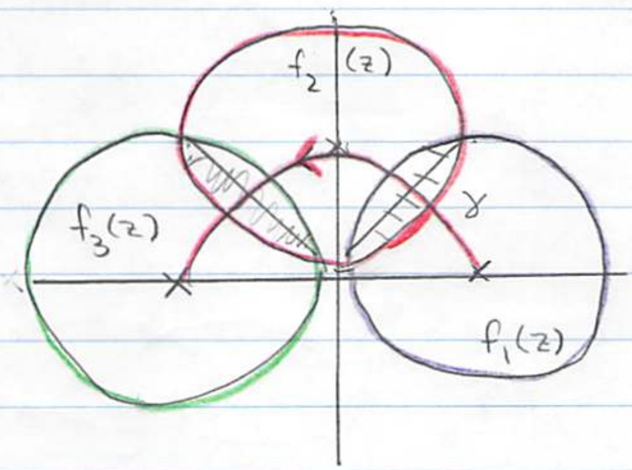
Suppose I just started with the $f_1(z)$:



- $f_1(z)$ is a function defined on the disk around $z=1$
- $f_2(z)$ is a function defined on the disk around $z=i$

The value of $f_1(z)$ on the line, determines uniquely the pow-series coefficients at $z=i$ giving rise to $f_2(z)$. We have therefore extendended the domain of definition of $f_1(z)$ through this process of demanding agreement in the overlapp

- Then, $f_2(z)$ could be used to define a power series around $z=-1$, $f_3(z)$



$$\frac{1}{z} = \frac{1}{z+1-1} \Rightarrow$$

$$f_3(z) \equiv -1 + (z+1) - (z+1)^2 + \ldots$$

- So from $f_1(z)$ defined at $z=1$, I have determed uniquely a function $f_3(z)$, defined near $z=-1$, by demanding that the chain of functions $f_1(z)$, $f_2(z)$, $f_3(z)$ agree in their overlap regions. This is analytic continuation

along the arc γ (see figure above).
The way we have done this it is clear that $f_1(z)$, $f_2(z)$, $f_3(z)$ are just representations of the same analytic function, $1/z$. We have analytically continued $f_1(z)$ along an arc γ

Ex 2

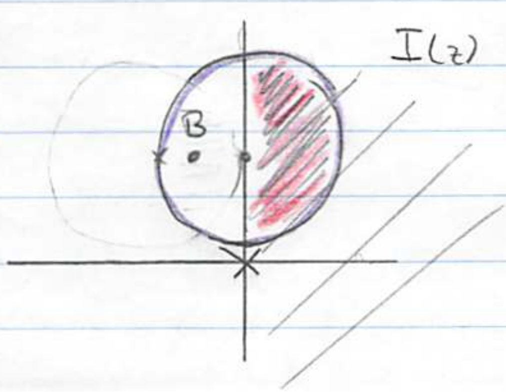
$$I_0(z) \equiv \int_0^\infty e^{-zt}\,dt$$

- This defines a function of z for $\mathrm{Re}\, z > 0$, much the way $f_1(z)$ defined a function on the disk in the previous example.

- Doing the integral

$$I_0(z) = \frac{1}{z} \longleftarrow$$ this defines an analytic continuation of $I_0(z)$ for $\mathrm{Re}\, z < 0$

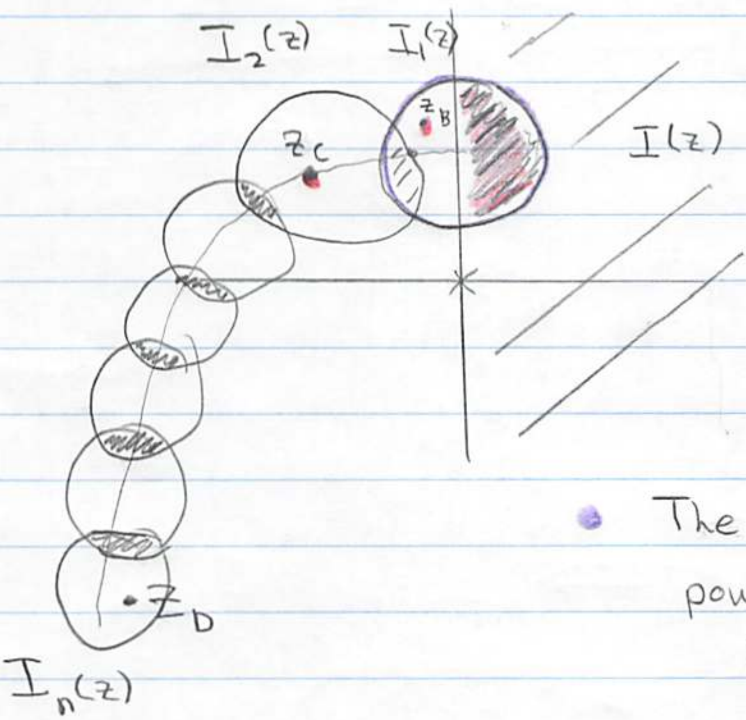
- Should you be worried about using the $1/z$ result for $\mathrm{Re}\, z$ negative? No you shouldn't. $1/z$ agrees with $I(z)$ for $\mathrm{Re}\, z > 0$. Specifically:



- $I(z)$ determines a series series at $z = i$ by uniquely fixing the coefficients of the $z = i$ power series from the $\mathrm{Re}\, z > 0$ results where $I(z)$ is defined.

- The value at the point B (where $I_o(z)$ is not defined) is then determined (uniquely) by this series. This power series must return $\frac{1}{z_B}$, to be in agreement with $I(z)$ for $Re\, z>0$

- We can continue the process along any path:



- The power series $I_2(z)$ must return $1/z_C$ in order that it agrees with $I_1(z) = 1/z$ in their overlapp region

- The process is repeated -- The power series $I_n(z_D)$ returns $1/z_D$

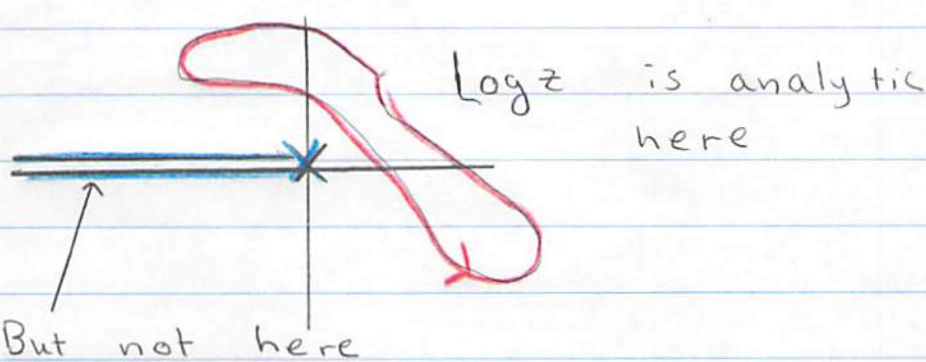
The Logarithm

$$\log z = \log re^{i\theta}$$
$$= \log r + i\theta$$

- But θ is not determined precisely since we can always add 2π to it. We could define the principal branch of the log $\equiv \mathrm{Log}(z)$

note capitol letter → $\mathrm{Log}(z) = \log r + i\theta \qquad -\pi < \theta \leq \pi$

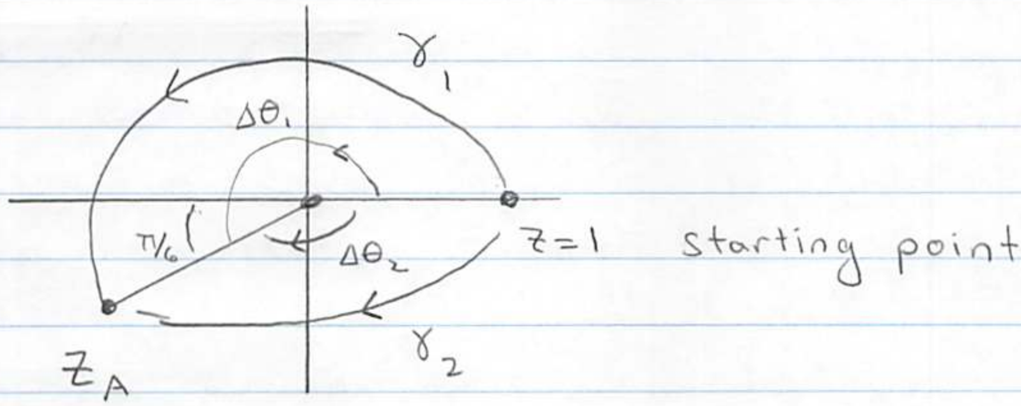
- This is fine, but then our function is not continuous across the negative axis and is therefore not analytic there



- More generally define the logarithm along a path γ:

$$\log_\gamma(z) = \int_1^z \frac{dz'}{z'} \qquad \frac{dz}{z} = \frac{dr}{r} + i\,d\theta$$

$$= \log r + i\,\Delta\theta \longleftarrow \underline{\text{Change}} \text{ in } \theta_f - \theta_i$$



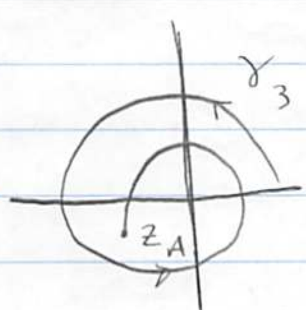
Then for the point z_A (reached by path γ_1)

$$\log_{\gamma_1} z_A = \log r + i(\pi + \pi/6)$$

While for the point z_A (reached by path γ_2)

$$\log_{\gamma_2} z_A = \log r - i(\pi - \pi/6)$$

The two logarithms differ by 2π since the combined path winds around the origin once.
Similarly take the path γ_3



Then

$$\log_{\gamma_3}(z) = \log r + i(3\pi + \pi/6)$$

The Log and analytic continuation

We can equivalently define the log through analytic continuation along an arc (See figure!)

The figure is given on the next page: study it !

- We start at point A on the real axis. We demand it equals $\log x_A$. The power series on the real axis:

$$\log(x) = \log(1 + (x-1))$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \dots$$

Replacing x with z gives a function defined on the disc which returns (see figure)

$$\log z = \log r + i\theta \qquad -\frac{\pi}{12} < \theta < \pi/12$$

without ambiguity in order to agree @ $\log x_A$ on the real axis.

- The next disc on the blue arc gives (to agree with the first disc in overlapp region)

$$\log z = \log r + i\theta \qquad \text{with} \qquad \frac{\pi}{12} < \theta < \frac{3\pi}{12}$$

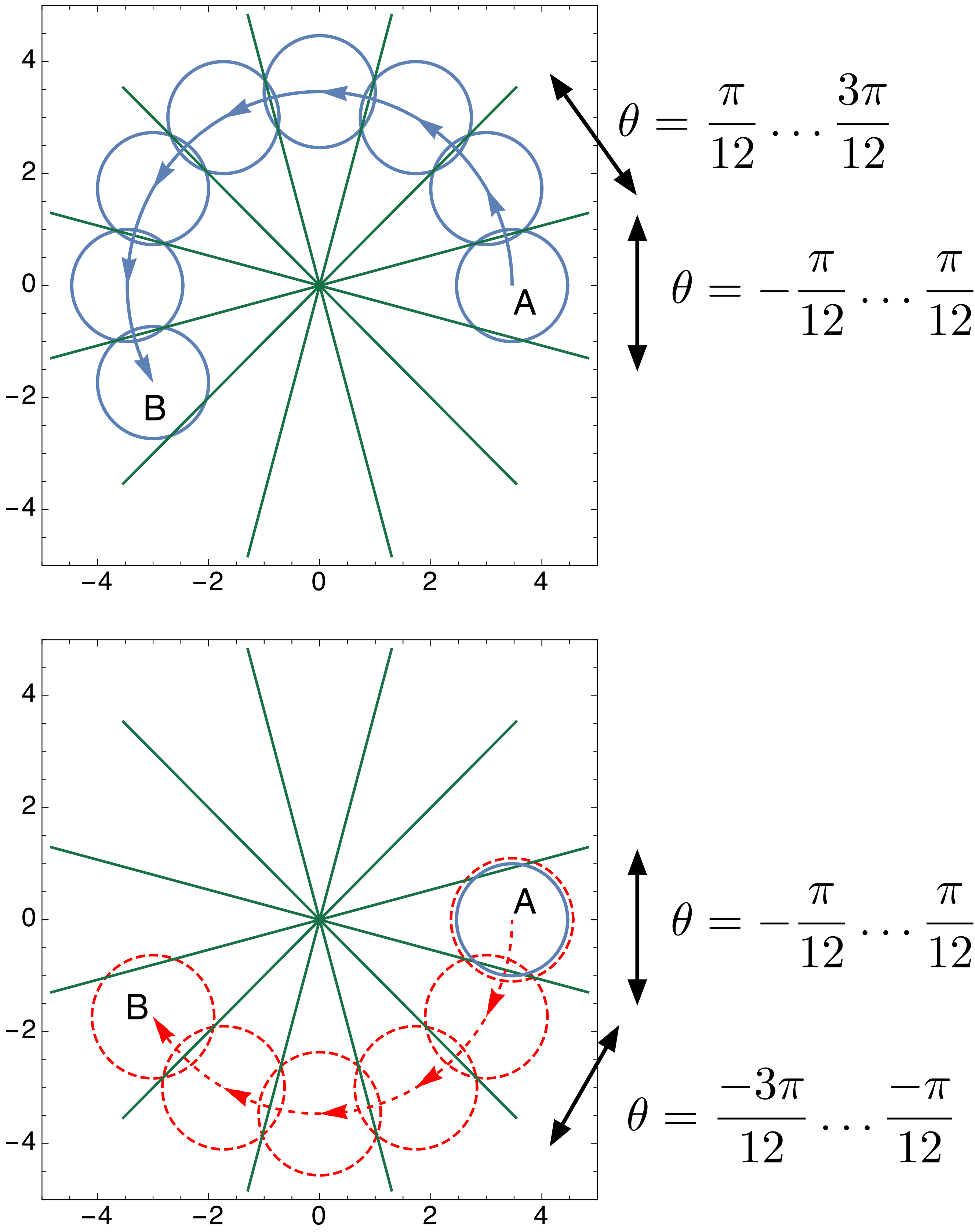
Finally, we reach point B and the $\log z$ defined through this process of analytic continuation on the arc γ_1 gives without ambiguity

$$\log_{\gamma_1} z_B = \log r + i(\pi + \pi/6)$$

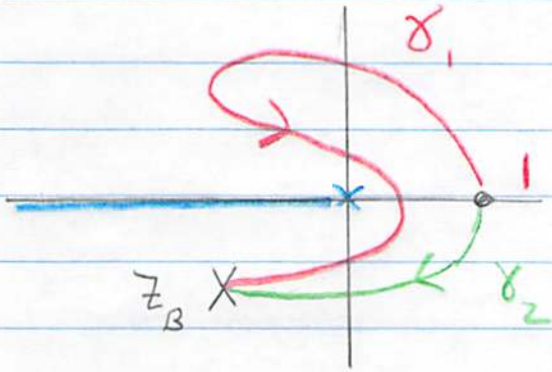
- Similarly for γ_2, the analytic continuation gives

$$\log_{\gamma_2} z_B = \log r - i(\pi - \pi/6)$$

Analytic continuation of the logarithm along two paths from A to B, γ_1 and γ_2. The first analytic continuation returns $\log_{\gamma_1}(z_B) = \log(r_B) + i(\pi + \frac{\pi}{6})$. The second returns $\log_{\gamma_2}(z_B) = \log(r_B) + i(-\pi + \frac{\pi}{6})$



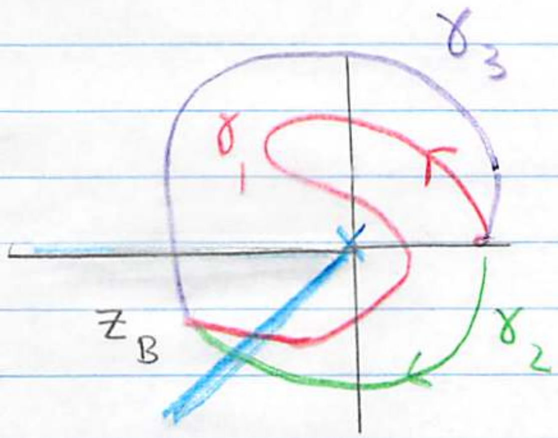
To keep apart the possible values of the logarithm, we draw a conventional line from the singularity to $-\infty$ known as a cut



Now any two curves which do not cross the cut, have not when taken together encircled the origin.

They therefore give the same analytic continuation
$\log_{\gamma_1}(z_B) = \log_{\gamma_2}(z_B) = \log r - i(\pi - \pi/6)$.

The cut is convention;



γ_1 and γ_2 will give the same as above

$$L_{\gamma_1} = L_{\gamma_2} = \log - i(\pi - \pi/6)$$

They will differ from γ_3.

The fact that γ_1 and γ_2 cross the cut, and γ_3 doesn't means they have together (γ_3, γ_1) encircled the origin once

Here I mean the close path (g3- g1) or (g3 - g2)

$$L_{\gamma_3} = \log r + i(\pi + \pi/6)$$

The Power Function

Given the $\log_\gamma(z)$ we define

$$z^\alpha = e^{\alpha \log_\gamma(z)} = r^\alpha e^{i\alpha\theta_\gamma}$$

angle depends on path

The power function is defined by a starting point and path. If two paths wind around the origin they will give different answers

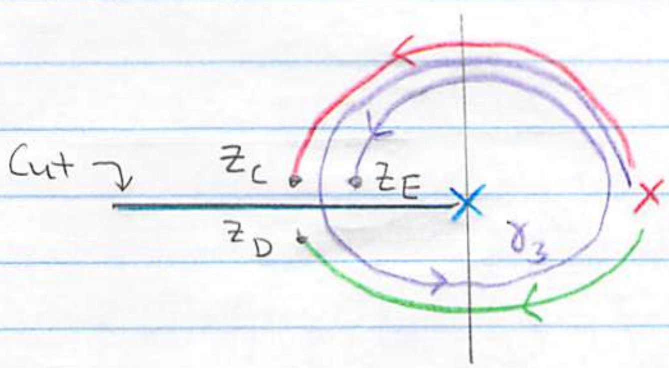
Take $\sqrt{z} \equiv S_\gamma(z)$ ← here we notate the path.

Usually the path is not notated, and it is understood that the path avoids a conventional branch cut, e.g.:

$$\sqrt{z_C} = \sqrt{r}\, e^{i\theta_C/2} = \sqrt{r}\, i$$

$$\sqrt{z_D} = \sqrt{r}\, e^{-i\theta_D/2} = \sqrt{r}\,(-i)$$

i.e. $\theta_C = \pi$ $\theta_D = -\pi$



But for $\sqrt{z_E} = \sqrt{r}\, e^{i\theta_E/2} = \sqrt{r}\, e^{i3\pi/2} = \sqrt{r}\,(-1)\, e^{i\pi/2}$

$$= \sqrt{r}\,(-i)$$

Here we should probably notate the path, $\theta_E = 3\pi/2$

$$\sqrt{z_E} \equiv S_{\gamma_3}(z)$$