Principal Value Integrals

- Consider the integral

$$I_{PV} = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi \leftarrow \text{Let's show this.}$$

- Want to use complex analysis. Sin and Cos by themselves dont behave as well as e^{ix}

$$I_{PV} = \frac{1}{i}\int_{-\infty}^{\infty} dx \left[\underset{\text{even}}{\cos x} + \underset{\text{odd}}{i\sin x} \right] \underset{\text{odd}}{\frac{1}{x}}$$

- The integral of an (even times odd) function is zero.

- The only problem is $\int \frac{\cos x}{x}\, dx$ is divergent

- Define the Principal Value integral / function sequence

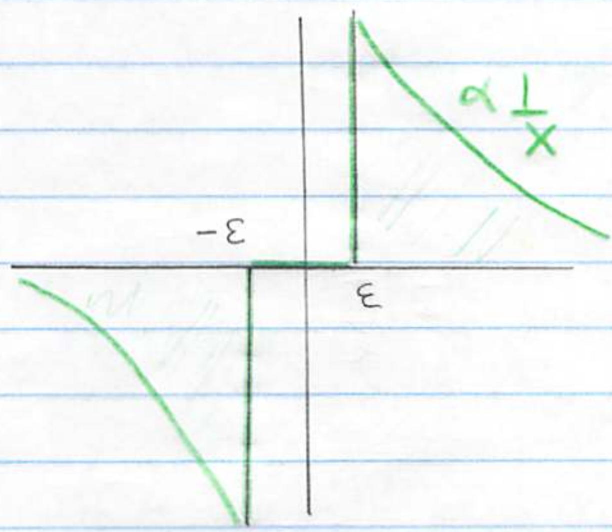
$$\int_{-\infty}^{\infty} dx\, PV\left(\frac{1}{x}\right) \equiv \lim_{\varepsilon \to 0} \int_{-\infty}^{-\varepsilon} \frac{1}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{1}{x}\, dx = 0$$

- You should understand that

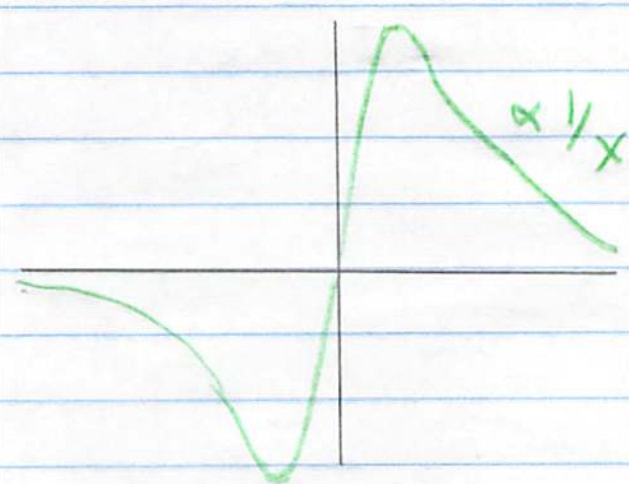
$$PV\left(\frac{1}{x}\right) \equiv \frac{P}{x} \quad \text{is a sequence of functions}$$

which as $\varepsilon \to 0$, approaches $1/x$ everywhere, except the origin where it is an odd function

- Here is one representation of $PV_\varepsilon(1/x)$ for a finite ε, that we used on the previous page:



- Here is another that we will use:



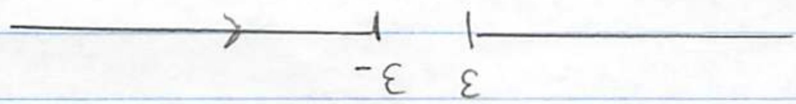
$$PV\left(\frac{1}{x}\right) = \lim_{\varepsilon \to 0} \frac{x}{x^2+\varepsilon^2}$$

its $1/x$ everywhere except right near the origin.

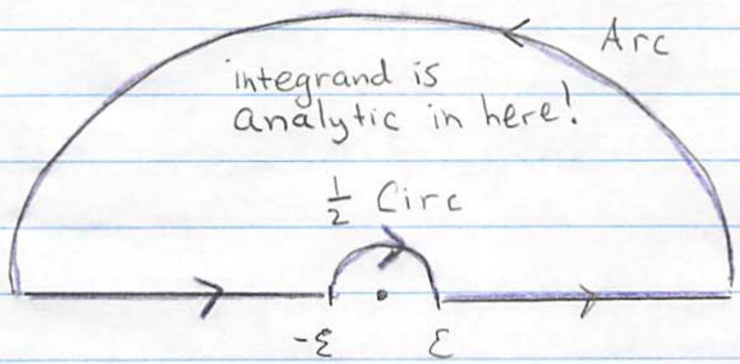
Then

$$I_{PV} \equiv \frac{1}{i}\int_{-\infty}^{\infty} e^{ix}\, PV\left(\frac{1}{x}\right) = \int_{-\infty}^{-\varepsilon} \frac{e^{ix}}{ix}\,dx + \int_{\varepsilon}^{\infty} \frac{e^{ix}}{ix}\,dx$$

i.e. we integrate



Now to turn I_{PV} into a closed contour in the complex plane we close the arc as follows



Convince yourself that the Arc at ∞ contribution vanishes exponentially as $|z| \to \infty$

Then

$$I_{contour} = \oint \frac{dz}{iz} e^{iz} = \underbrace{0}_{\text{since integrand is analytic}} = I_{PV} + \underbrace{I_{\frac{1}{2}circ}}_{\text{integral over 1/2 circ.}}$$

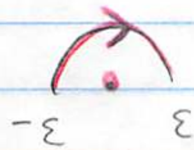
i.e.

$$I_{PV} = -I_{1/2\,circ}$$

Thus $$I_{pv} = - I_{1/2\ circ}$$

$$= - \int_{1/2\ circ} \frac{dz}{iz} e^{ikz}$$

For $1/2$ circ,



we have $e^{ikz}\Big|_{\text{near origin}} = 1$ and then

$$\frac{dz}{z} = -i d\theta, \quad \text{so} \quad \int \frac{dz}{z} = -i \int_0^{\pi} d\theta$$

And

$$I_{pv} = -\frac{1}{i} \left[\underbrace{(2\pi i)}_{\text{pole contib}} \underbrace{(-1)}_{\text{"wrong" way around pole}} \underbrace{\left(\tfrac{1}{2}\right)}_{1/2\ \text{arc}} \right]$$

$$I_{pv} = \int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi$$

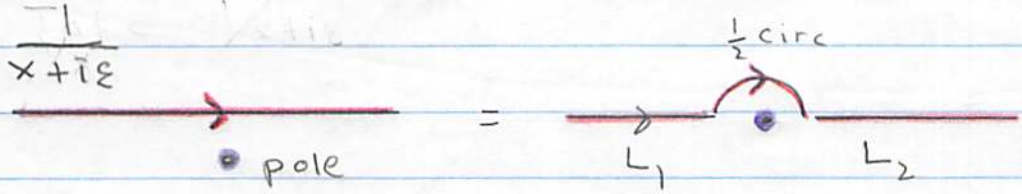
A common expression

$$\lim_{\varepsilon \to 0} \frac{1}{x \mp i\varepsilon} = \lim_{\varepsilon \to 0} \frac{x}{x^2+\varepsilon^2} + \frac{i\varepsilon}{x^2+\varepsilon^2}$$

$$= PV \frac{1}{x} \pm i\pi\, \delta(x)$$

You should understand this with complex analysis:

$\frac{\mp 1}{x+i\varepsilon}$



So

$$\int dz \frac{f}{z+i\varepsilon} = \int_{L1+L2} \frac{f(z)}{z} + \int_{1/2\, circ} \frac{f(z)}{z}$$

acts like a δ-fcn

$$= \left[\int dz\, f(z)\, PV \frac{1}{z} \right] - i\pi f(0)$$

or

$\int dz\, f(z)\, -i\pi\, \delta(z)$