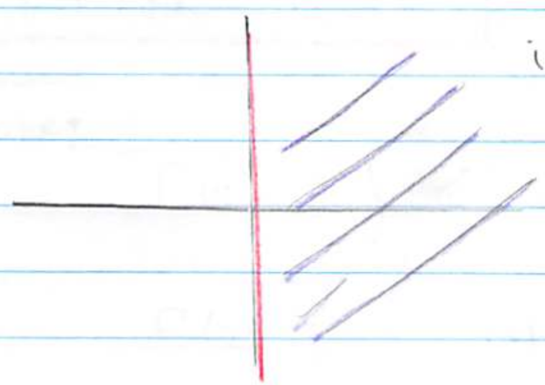
The Γ-fcn

The $\Gamma(z)$ function occurs all over in physics

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^z \, \frac{dt}{t}$$

① We write $t^z \frac{dt}{t} = t^{z-1}\,dt$ sometimes to emphasize that dt/t is invariant under a scale change $t \to t' = at$.

② This representation is defined for $\operatorname{Re} z > 0$



③ A change of variables $t = x^2$ yields

$$\Gamma(z) = 2\int_0^\infty dx\, e^{-x^2}\, x^{2z-1}\, dx$$

Showing that $\Gamma(z)$ records certain moments of Gaussiian Integrals

4 Integrating by parts we establish one of the most imortant properties

$$t^{z-1} = \frac{1}{z}\frac{dt^z}{dt}$$

So

$$z\Gamma(z) = \int_0^\infty dt\, e^{-t}\, \frac{dt^z}{dt}\, dt$$

$$= e^{-t} t^z \Big|_0^\infty - \int_0^\infty \overbrace{\frac{de^{-t}}{dt}}^{-e^{-t}} t^z\, dt$$

$$\boxed{z\Gamma(z) = \Gamma(z+1)}$$

5 Note:

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1 = 0!$$

$$\Gamma(2) = 1\,\Gamma(1) = 1 = 1!$$

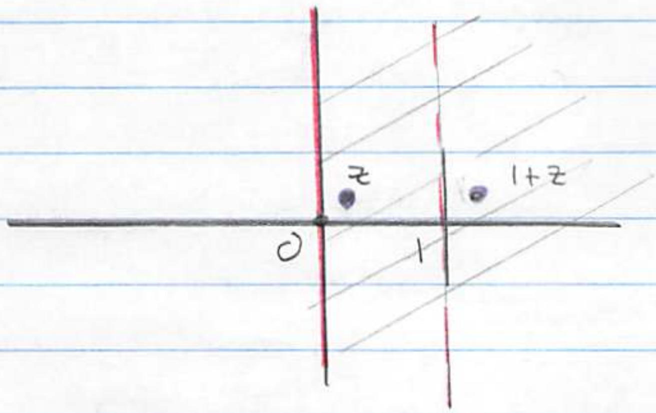
$$\Gamma(3) = 2\,\Gamma(2) = 2!$$

$$\Gamma(4) = 3\cdot\Gamma(3) = 3! \quad \text{etc} \qquad \boxed{\Gamma(n+1) = n!}$$

6 Note: $\boxed{\Gamma(1/2) = 2\int_0^\infty dx\, e^{-x^2} = \sqrt{\pi}}$

So $\Gamma(5/2) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

Analytic Continuation to $\operatorname{Re} z < 0$



Take $z > 0$ but small close to origin

$$\Gamma(z+1) = z\,\Gamma(z)$$

So

$$\frac{\Gamma(z+1)}{z} = \Gamma(z)$$

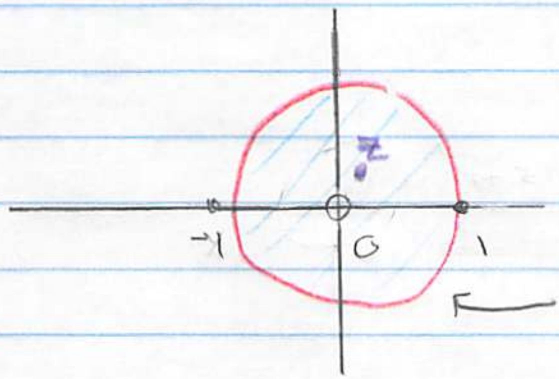
Now $\Gamma(z+1) = \int_0^\infty dt\, e^{-t}\, t^z$ is perfectly well behaved and has a convergent power series for $|z| < 1$.

$$\Gamma(z+1) = 1 + C_1 z + C_2 z^2 + \dots$$

So we see that $\Gamma(z)$ has a series expansion of the form:

$$\Gamma(z) = \frac{1}{z} + C_1 + C_2 z + \dots$$

Thus we see that $\Gamma(z)$ has a simple pole at the origin, but otherwise has a conergent (Laurent) series in a puctured disk around $z=0$. This provides an analytic continuation of the Γ-fcn to negatize z



the power series converges $|z|<1$

- What this means in practice is that for $-1<z<0$ we write $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ and evaluate $\Gamma(z+1)$ using the integral.

A side remark:

$$\Gamma(1+z) = \int_0^\infty dt\, e^{-t}\, t^z$$

$$= \int_0^\infty dt\, e^{-t}\left[1 + z\log t + \frac{z^2}{2!}(\log t)^2 + \ldots\right]$$

$$= 1 \underbrace{-\gamma_E}\, z + O(z^2)$$

the coefficient C_1 on previous page has a name!

- Where

$$-\gamma_E \equiv \int_0^\infty dt\, e^{-t}\log t = -0.577\ldots$$ is known

as the Euler-Mascheroni constant, $\Gamma(z) \simeq \frac{1}{z} - \gamma_E \quad |z| \ll 1$

The process can be continued. Let n be the smallest integer such that $z+n > 0$.
Then

$$\Gamma(z+n+1) = (z+n)(z+n-1)(z+n-2)\dots\dots z\,\Gamma(z)$$

And

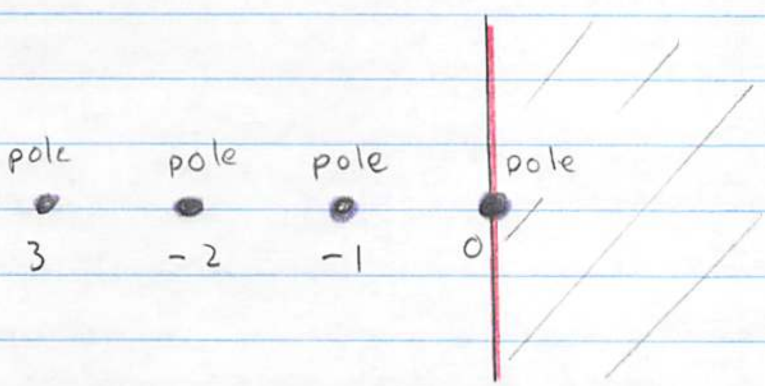
$$\Gamma(z) = \frac{\Gamma(z+n+1)}{(z+n)(z+n-1)(z+n-2)\dots z}$$

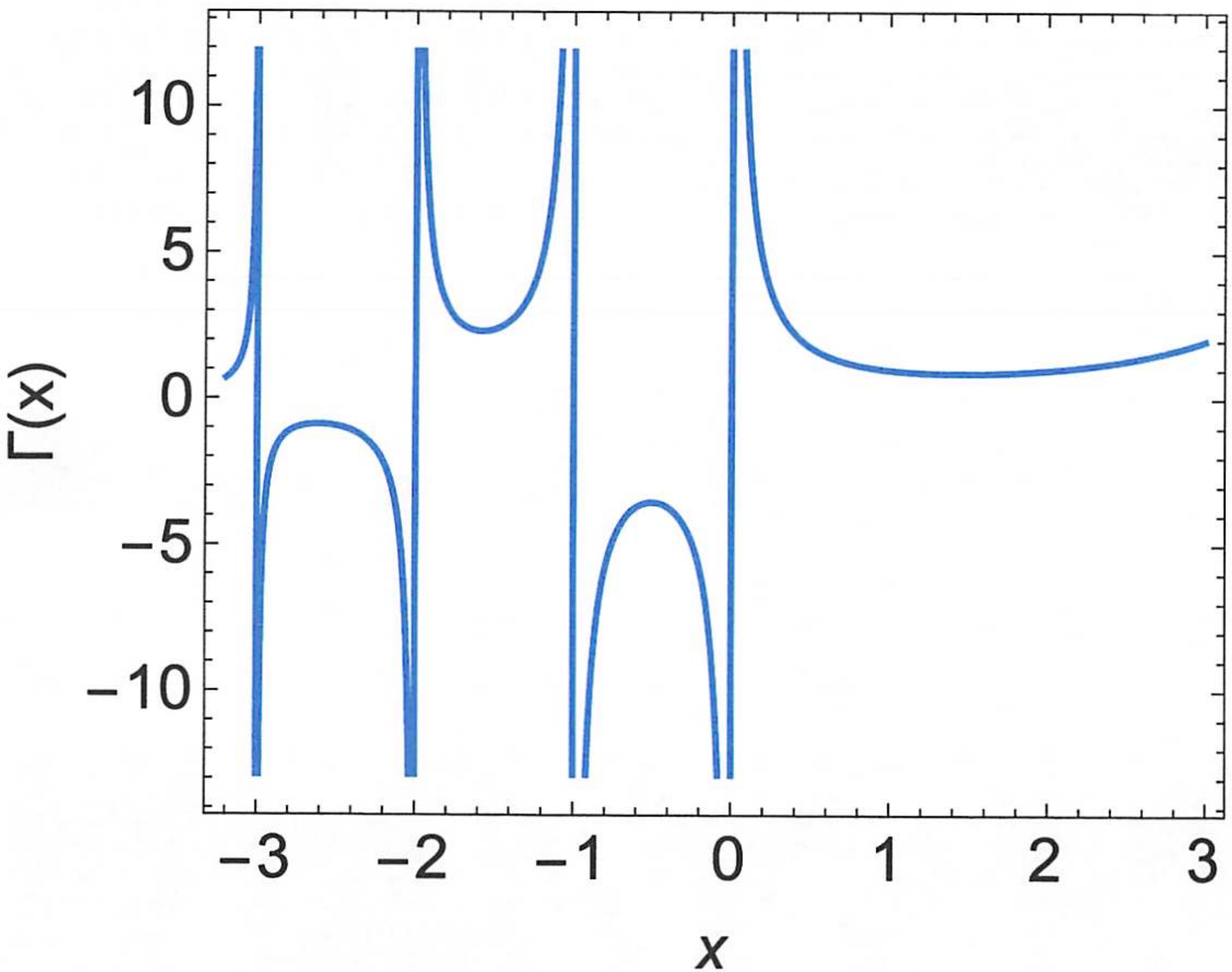
So for z near a negative integer $z=-n+\varepsilon$ we have

$$\Gamma(-n+\varepsilon) = \frac{\Gamma(1+\varepsilon)}{\underset{\substack{\uparrow\\ \text{pole}}}{\varepsilon}\,(\varepsilon-1)(\varepsilon-2)\dots\dots(\varepsilon-n)}$$

Thus there are poles at all negative integers
Taking $\varepsilon \to 0$ we see (since $\Gamma(1)=1$) that

$$\Gamma(-n+\varepsilon) \simeq \frac{1}{\varepsilon}\frac{(-1)^n}{n!} \longleftarrow \text{the residues are } (-1)^n/n!$$





Note poles alternating in sign and decreasing in strength, for negative integers and zero.

Connection to Gaussian Integrals $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

- Cleary from the definition

$$\Gamma(z) = 2\int_0^\infty dx\, e^{-x^2}\, x^{2z-1}$$

$\Gamma(z)$ is related to Gaussian Integrals.

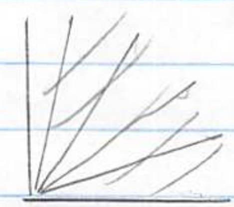
- Let us recall how $2\int_0^\infty dx\, e^{-x^2} = \sqrt{\pi}$ is proved

$$\Gamma(1/2) = 2\int_0^\infty dy\, e^{-y^2}$$

$$\Gamma(1/2) = 2\int_0^\infty dx\, e^{-x^2}$$

So

$$\Gamma(1/2)^2 = 4\int_0^\infty\int_0^\infty \overbrace{dx\,dy}^{r\,dr}\, \overbrace{e^{-x^2+y^2}}^{e^{-r^2}}$$



$$\left(\Gamma(1/2)\right)^2 = \underbrace{\left[2\int_0^\infty r\,dr\, e^{-r^2}\right]}_{1}\underbrace{\left[2\int_0^{\pi/2} d\theta\right]}_{\pi}$$

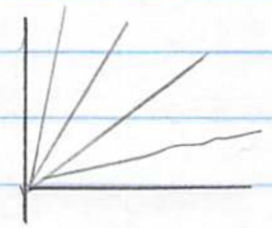
- Gives $\Gamma(1/2) = \sqrt{\pi}$

- Now we will general this neat trick by Gauss

The Generalization #1

$$\Gamma(z) = 2\int_0^\infty dy\, e^{-y^2}\, y^{2z-1}$$

$$\Gamma(1-z) = 2\int_0^\infty dx\, e^{-x^2}\, x^{-(2x-1)}$$



So

$$\Gamma(z)\,\Gamma(1-z) = 4\int_0^\infty dx \int_0^\infty dy\; e^{-x^2+y^2} \left(\frac{y}{x}\right)^{2z-1}$$

Now change vars $dx\,dy = r\,dr\,d\theta$ $\quad y/x = \tan\theta$

$$\Gamma(z)\,\Gamma(1-z) = \underbrace{\left(2\int_0^\infty r\,dr\, e^{-r^2}\right)}_{1} \underbrace{\left(2\int_0^\pi (\tan\theta)^{2z-1}\, d\theta\right)}_{I_\theta(z)}$$

The remaining trig integral is done as follows
let

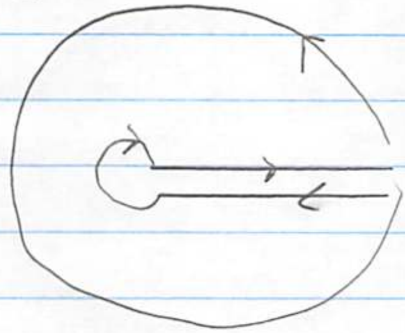
$$u = \tan\theta \qquad \frac{du}{1+u^2} = d\theta \qquad \left(\text{think } \int \frac{1}{1+u^2} = \tan^{-1}\right)$$

So

$$I_\theta(z) = 2\int_0^\infty \frac{u^{2z-1}}{1+u^2}\, du$$

Happily we did this integral just a few days ago, ☺! We did it by considering a contour

remember this!



$I_\Theta(z) = \dfrac{\pi}{\sin \pi z}$, Check for $z = 1/2$ $I_\Theta(z) = 2\int_0^\infty \dfrac{du}{1+u^2}$

$= \pi$

Thus we have established an important identity

$$\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}}$$

We will have more on this result shortly

The "β" function $B(p,q)$ - The Generalization #2

Let us consider the "Gauss Trick" a bit further

Define angular integrals of a very general sort

$$B(p,q) \equiv 2\int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}\, d\theta$$

We can use the Gauss-Trick to express these integrals in terms of the Γ-fcn

$$\Gamma(p) = 2\int_0^\infty dy\, e^{-y^2} \overbrace{r^{2p-1}(\sin\theta)^{2p-1}}^{y^{2p-1}} \qquad y = r\sin\theta$$

$$\Gamma(q) = 2\int_0^\infty dx\, e^{-x^2} \overbrace{r^{2q-1}(\cos\theta)^{2q-1}}^{x^{2q-1}} \qquad x = r\cos\theta$$

Combining these two with the Gauss Trick

$$\Gamma(p)\Gamma(q) = \overbrace{\left[2\int r\,dr\, e^{-r^2} r^{2p-1} r^{2q-1}\right]}^{=\Gamma(p+q)}$$

$$\times \left[2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\right]$$

Thus we find

$$\boxed{\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p,q)}$$

The $B(p,q)$ integral can be expressed several other useful ways. Set $t = \sin^2\theta$ $\quad \frac{dt}{t(1-t)} = \frac{2\,d\theta}{\sin\theta\cos\theta}$

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$$