Linear Differential Equations

- Take a formal operator:

$$\mathcal{L} = Q(x) + \mathbb{P}(x) \frac{d}{dx} + \frac{d^2}{dx^2} \qquad \text{(Eq 1)}$$

Then

inhomogeneous equation

$$\mathcal{L}\, y(x) = f(x)$$

Source term

homogeneous equation

$$\mathcal{L}\, y(x) = 0$$

- We say the equation is homogeneous if there is no source term

- Generally it is best to study the homogeneous equation first and find its solutions, then turn to its inhomogeneous counterpart

Broad Classes of Equations: intial value & Boundary Value

- Initial value problems - easier

Harmonic oscillar

$$\left\{ \frac{d^2 y}{dt^2} + \omega_0^2 y = f(t) \qquad y(t)\Big|_{t_0} = y_0 \qquad \dot{y}\Big|_{t_0} = V_0 \right.$$

- Boundary value problems

particle in a box

$$\left\{ \frac{d^2 y}{dx^2} + k^2 y = 0 \qquad y(0) = 0 \quad y(L) = 0 \right.$$

- Boundary value problems are invariably harder to solve since one is interested in a more global character, i.e. how the function behaves at two ends.

Homogeneous Boundary Conditions

- We consider a class of functions (not necessarily solutions!) which are defined on $[a,b]$ with certain boundary conditions. We say that the boundary conditions are homogeneous if given $y_1(x)$ and $y_2(x)$ satisfying the b.c. then so does

$$c_1 y_1(x) + c_2 y_2(x)$$

- The functions satisfying homogeneous b.c. form a linear vector space. For $p_0(x)$ and $p_1(x)$ regular, the most general form of homogeneous boundary conditions is

$$\alpha_{11}\, y(a) + \alpha_{12}\, y(b) + \beta_{11}\, y'(a) + \beta_{12}\, y'(b) = 0$$

$$\alpha_{21}\, y(a) + \alpha_{22}\, y(b) + \beta_{21}\, y'(a) + \beta_{22}\, y'(b) = 0$$

any two of these coefficients (one per line e.g. $\alpha_{11} = \alpha_{21} = 1$) can be set to unity.

Examples

(1) Take particle in box $y'' + k^2 y = 0$. The b.c. are homogeneous and there are two of them

$$y(0) = 0 \qquad y(L) = 0$$

(2) Harmonic Oscillator with Specific inital conditions

$$y(t)\Big|_{t_0} = y_0 \qquad \dot{y}(t)\Big|_{t_0} = V_0$$ ← these are inhomagen boundary conditions

(3) Driven harmonic oscillator which isn't moving before the $\int$ tarts

$$\ddot{x}(t) + \omega_0^2 x(t) = f(t)$$

With $\begin{cases} x(t) = 0 \quad \text{for all } t < t_0 \\ \text{This is a homogeneous "causal" boundary condition} \end{cases}$

(4) For first order equations, e.g.

$$-i \frac{d\psi}{dx} = p\psi$$

Only one b.c. needs to be specified

$$\psi(0) = \psi(L) \Leftarrow \begin{cases} \text{homogeneous} \\ \text{"periodic" boundary conditions} \end{cases}$$

Sturm Liouiville form

- Take a general second order differential equation of a slightly different form

$$\left[-\frac{d}{dx}\,p(x)\,\frac{d}{dx} + q(x)\right] y = 0 \qquad (4.1)$$

This is known as Sturm-Liouville form. Differentiating through and dividing by $-p(x)$

- $$\left[\frac{d^2}{dx} + \frac{p'(x)}{p(x)}\frac{d}{dx} - \frac{q(x)}{p(x)}\right] y = 0$$

This is the same as Eq.(1) with $\mathbb{P}(x) = \dfrac{p'(x)}{+p(x)}$ and $Q(x) = -q(x)/p(x)$. Specifially

$$\frac{d \ln p(x)}{dx} = \mathbb{P}(x)$$

$$p(x) = C \exp\left[+\int^{x} dx' \, \mathbb{P}(x')\right] \qquad (4.2)$$

- Most 2nd order DEQ equations in physics are more naturally expressed in second order form

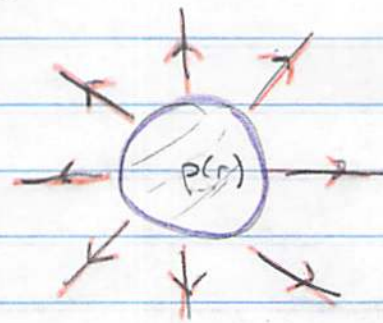
- For instance take the Laplace Equation:

potential

$$-\nabla^2 \Phi = \rho(r)$$

with spherically symmetric charge density

$$-\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$



field lines.

Then $\Phi(r)$ satisfies

$$-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\Phi(r) = \rho(r)$$

- We will later see that Eq (4.1) is essentially the most general form of a self adjoint second order differential operators in 1D.

The Wronskian

- The most general of solution of the second order solution to the DEQ is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

Where c_1 and c_2 are constants and $y_1(x)$ and $y_2(x)$ are linearly independent.

- The Wronskian can let you know is the solutions are linearly independent

Specifically given an initial condition

$$y(x)\Big|_{x=x_0} = y_0 \quad \text{and} \quad \frac{dy}{dx}\Big|_{x=x_0} = y'_0$$

We want to adjust c_1 and c_2 to match the initial conditions

$$y_0 = c_1\, y_1(x_0) + c_2\, y_2(x_0)$$

$$y'_0 = c_1\, y'_1(x_0) + c_2\, y'_2(x_0)$$

Eq (6.0)

This has solutions provided $\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} \equiv W(x_0)$

is not zero.

We will now show that (Abel's Formula)

$$\boxed{\begin{aligned} &p(x)\, W(x) = \text{constant} \\ \text{or}\quad &W(x) = \exp\left(-\int^x \mathbb{P}(x')\,dx'\right) \end{aligned}} \qquad (6.1)$$

Proof:

$$\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right]y_1(x) = 0 \qquad (7.1)$$

$$\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right]y_2(x) = 0 \qquad (7.2)$$

Multiply the first equation by $y_2(x)$ and the second equation by $y_1(x)$ and subtract

$$y_1\frac{d}{dx}p(x)\frac{d}{dx}y_2 - y_2\frac{d}{dx}p(x)\frac{d}{dx}y_1(x) = 0$$

Or

$$\frac{d}{dx}\left[p(x)\,y_1(x)\,y_2'(x) - p(x)\,y_2(x)\,y_1'(x)\right] = 0$$

i.e.

$$p(x)\left[y_1(x)y_2'(x) - y_2(x)y_1'(x)\right] = \text{Const} \qquad (7.3)$$

- The meaning of such wronksian equations is that flux is conserved (more shorly)

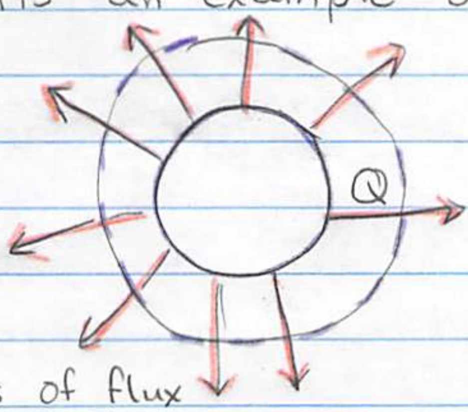
- From a perspective of the solutions to the differential equation, Abel's formula tells us ...

... tells us that our general solution in one region, remains the general solution in any region which can be reached without passing through the zeros of $p(x)$, where

$$W(x) = \frac{\text{Const}}{p(x)} \leftarrow \text{singular at zeros of } p(x)$$

is singular.

- As an example of Wronskinan and Flux. Just consider outside a charged sphere:



$$-\frac{d}{dr} r^2 \frac{d}{dr} \Phi(r) = 0$$

This has solutions $\underset{y_1(r)}{1}$ and $\underset{y_2(r)}{\frac{1}{r}}$.

Then $p(r) = r^2$

$$W(r) = y_1(r) y_2'(r) - y_2(r) y_1'(r)$$

$$= \frac{\partial}{\partial r} \frac{1}{r}$$

- Thus we find

$$p(r)\, W(r) = \underbrace{r^2}_{\text{area}} \underbrace{\frac{\partial}{\partial r} \frac{1}{r}}_{\text{electric field}} = -1 = \text{constant}$$

constant: area x electric field = electric flux

First order equations   A General Solution

- Many of the steps leading to the Sturm-Liouville form can be used to solve first order inhomogeneous Eq's in complete generality

- Consider

$$\frac{dV}{dt} + \eta(t) V(t) = f(t)$$

This describes the velocity of a particle subjected to a time dependent drag coefficient and external force.

- Motivated by the algebra

$$\frac{1}{p(t)} \frac{d}{dt} \left(p(t) V(t) \right) = \dot{V} + \frac{\dot{p}}{p} V$$

We set

$$\frac{\dot{p}}{p} = \frac{d}{dt} \left(\ln p(t) \right) = \eta(t)$$

and find $\boxed{p(t) = \exp \left(\int_0^t dt' \, \eta(t') \right)}$

This is known as an integrating factor

Thus, we find a simpler equation

$$\frac{1}{p(t)} \frac{d}{dt}\left(p(t) \frac{dv}{dt} \right) = f(t)$$

- First consider the homogeneous solution $f(t) = 0$
Then

$$\frac{d}{dt}\left(p(t) \frac{dv}{dt} \right) = 0$$

And

$$v = \frac{C}{p(t)}$$

$$v(t) = C \exp\left[-\int_0^t \eta(t')\,dt' \right]$$

For constant drag $v(t) = e^{-\eta t}\, C$.

- For this case it is simple enough to solve the inhomogeneous equation

$$\frac{d}{dt} p(t)\, v(t) = p(t) f(t)$$

$$p(t)\, v(t) = \int^t d\underline{t}\; p(\underline{t})\, f(\underline{t}) + C$$

or

(10.1) $$\boxed{v(t) = \frac{C}{p(t)} + \frac{1}{p(t)} \int^t d\underline{t}\; p(\underline{t})\, f(\underline{t})}$$

General solution of 1st order Eq

We see that the general Solution is

$$V(t) = C\, e^{-\int^{t} \eta(t')\,dt'} + e^{-\int^{t} \eta(t')\,dt'} \left[\int^{t} d\underline{t}\; f(\underline{t})\, e^{+\int^{\underline{t}} \eta(t')\,dt'} \right]$$

- For constant η we have

$$V(t) = C e^{-\eta t} + \int^{t} f(\underline{t})\, e^{+\eta(\underline{t} - t)}\, d\underline{t}$$

- Take a specific initial inhomogeneous initial condition $V(t)\big|_0 = V_0$ then for $t \geq 0$

$$V(t) = \underbrace{V_0 e^{-\eta t}}_{} + \underbrace{\int_0^{t} f(\underline{t})\, e^{\eta(\underline{t} - t)}\, d\underline{t}}_{}$$

a solution to homogeneous equation $\mathcal{L}y = 0$ satisfying the inhomogeneous b.c.

$$V(t)\Big|_{t_0} = V_0$$

A particular solution satisfying $\mathcal{L}y = f(t)$ ← inhomo equation with homogeneous boundary condition

$$V(t)\Big|_{t_0 = 0} = 0$$

- This is always the structure of general solutions as we show next