So

$$(26.1) \quad G(t,t_0) = \frac{e^{-\eta/2(t-t_0)}}{m\omega_0} \sin(\omega_0(t-t_0)) \quad \theta(t-t_0)$$

The general solution is then

$$y(t) = A_1 e^{-\eta/2 t} e^{+i\omega_0 t} + A_2 e^{-\eta/2 t} e^{-i\omega_0 t}$$
$$+ \int_a^t f(t_0) \frac{e^{-\eta/2(t-t_0)}}{m\omega_0} \sin(\omega_0(t-t_0))\, dt_0$$

Causal Green Functions via Fourier Transform

- For very simple equations like the damped SHO or the drag equation the F.Transfrorm is an easy way to procede

$$\frac{dv}{dt} + \eta v = f(t) \qquad v(a) = V_0$$

- with homogeneous eq $dV/dt + \eta V = 0$, and Green function satisfying

$$\frac{dG}{dt} + \eta G = \delta(t-t_0) \qquad G(t=a, t_0) = 0 \qquad (26.2)$$

- For the homogeneous equation we look for solutions of the form $e^{-i\omega_* t} = 0$, since the equation has constant coefficients. Substituting into homog equation

$$(-i\omega_* + \eta)e^{-i\omega_* t} = 0 \quad \text{or} \quad \omega_* = -i\eta \qquad V(t) = Ce^{-\eta t}$$

- Now lets fourier transform Eq (26.2), $G(t) \to \hat{G}(\omega)$:

$$\frac{dG}{dt} \longrightarrow -i\omega \hat{G}(\omega)$$

$$\delta(t - t_0) \longrightarrow e^{+i\omega t_0}$$

Then Eq(26.2) in Fourier Space is

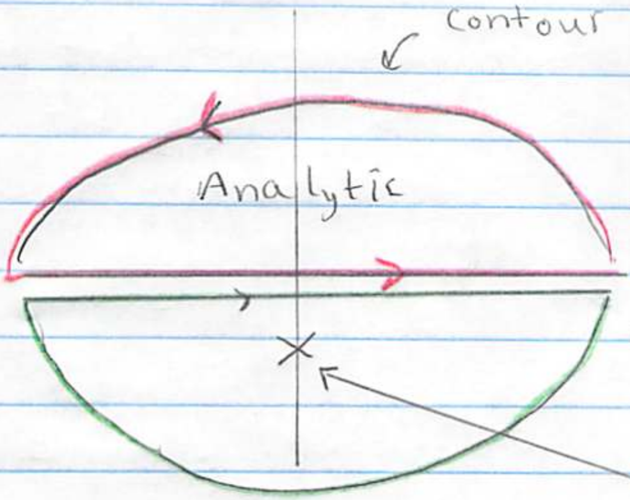
$$-i\omega\hat{G} + \eta\hat{G} = e^{i\omega t_0} \implies \hat{G} = \frac{e^{i\omega t_0}}{-i\omega + \eta}$$

- So by inverse FT

$$G(t,t_0) = \int_{-\infty}^{\infty} e^{-i\omega t} \frac{e^{+i\omega t_0}}{-i\omega + \eta} \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} \frac{e^{-i\omega(t-t_0)}}{-i\omega + \eta} \frac{d\omega}{2\pi}$$

We do these integrals by contour integration contour for $t - t_0 < 0$



contour for $t - t_0 > 0$

pole at $\omega = \omega_* = -i\eta$

notice this is the frequency of the homogeneous solution

Find $\qquad e^{-i\omega_* t} = e^{-\eta t}$

$$G(t,t_0) = \begin{cases} 0 & t - t_0 < 0 \\ e^{-\eta(t-t_0)} & t - t_0 > 0 \end{cases}$$

analyticity / causality

Homework will explore this a bit farther with damped sho.

2nd Order Differential Equations with Constant Coeffs

- Just substitute $e^{-i\omega t}$ and solve for ω.

$$\frac{d^2x}{dt^2} + \eta \frac{dx}{dt} + \omega_0^2 x = 0$$

$$-\omega^2 + i\eta\omega + \omega_0^2 x = 0 \qquad \omega_\pm = -\frac{i\eta}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\eta}{2}\right)^2}$$

Generally have two roots, and two sols

$$e^{-i\omega_+ t} \quad \text{and} \quad e^{-i\omega_- t}$$

- But it can happen that $\omega_+ = \omega_-$ (e.g. at the transition from underdamped to damped). In this case the solutions are $-i\omega_- = -\eta/2$

$$e^{-i\omega_- t} \quad \text{and} \quad t e^{-i\omega_- t} \quad \text{or} \quad e^{-\eta t/2} \quad \text{and} \quad t e^{-\eta/2\, t}$$

- Just imagine that $\omega_+ = \omega_- + \Delta\omega$, for instance by slightly changing η. Then for any $\Delta\omega$

One solution and a second

$$y_1(t) \equiv e^{-i\omega_- t}, \qquad y_2(t) \equiv \frac{e^{-i(\omega_- + \Delta\omega)t} - e^{-i\omega_- t}}{\Delta\omega} \longrightarrow \frac{d\, e^{-i\omega t}}{d\omega}$$

$$= -it\, e^{-i\omega_- t} \qquad \text{for } \omega_+ = \omega_-$$

Equidimensional (or Euler) Differential EQ

Notice that the "logarithmic" derivative is scale invariant. This means $x \to x' = ax$

$$x \frac{d}{dx} \to x' \frac{d}{dx'} = x \frac{d}{dx}$$

This suggests that for scale invariant differential equations

$$\left[c_1 x^2 \frac{d^2}{dx^2} + c_2 x \frac{d}{dx} + c_3 \right] y = 0$$

or

$$\left[x \frac{d}{dx} x \frac{d}{dx} + \bar{c}_2 x \frac{d}{dx} + \bar{c}_3 \right] y = 0 \qquad (30.1)$$

We should try a power law $y = x^s$ and solve for s. Substituting into (30.1) we have ~~to~~ solutions

$$\left[s^2 + c_2 s + c_3 \right] x^s = 0 \qquad s_\pm = -\frac{\bar{c}_2}{2} \pm \sqrt{\left(\frac{\bar{c}}{2}\right)^2 - c_3}$$

And then

$$y = C_1 x^{s_+} + C_2 x^{s_-}$$

these are branch point singularities in the complex plane

Here again the two roots can be degenerate

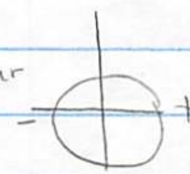
$$y_1 = x^{s_-} \qquad y_2 \equiv \frac{x^{s_- + \Delta s} - x^{s_+}}{\Delta s} \xrightarrow[\Delta s \to 0]{} \frac{d x^s}{d s}$$

$$= x^s \ln x \quad \text{for } \Delta s \Rightarrow 0$$

Example: Laplace eqn in cylindrical coords

$$-\nabla^2 \Phi = 0$$

assume this form for angular dependence



Then $\Phi = R_m(\rho) \overbrace{\cos m\phi}$

So the laplace equation dictates:

$$-\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right]\Phi = 0$$

Substitute $R = \rho^s$:

And So we must solve:

$$-s^2 + m^2 = 0$$

$$\left[-\rho\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + m^2\right] R_m = 0 \qquad s = \pm m$$

$$R_m = \begin{cases} C_1 \rho^m + C_2/\rho^m & m \neq 0 \\ C_1 + C_2 \ln \rho & m \to 0 \end{cases}$$

Classification of Points

$$\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x) = 0$$

① Regular point. If $P(x)$ and $Q(x)$ are analytic at x_0 the point is called a regular point and the solutions are analytic $y_1(x) = \sum a_n (x-x_0)^n$ there

② If $P(x)$ has a single pole $P(x) = \frac{p_{-1}}{x-x_0} + \ldots$ at x_0, and/or $Q(x)$ has a double pole at x_0:

$$Q(x) = \frac{q_{-2}}{(x-x_0)^2} + O\left(\frac{1}{x-x_0}\right)$$

The point x_0 is known as a regular singular point.

This is Fuch's theorem

The solutions will take the form, $\text{Re}\, s_1 \geq \text{Re}\, s_2$:

$$y_1 = x^{s_1} \text{ (analytic function)}$$

$$y_2 = x^{s_2} \text{ (analytic function)}$$

or usually if $s_1 - s_2 = 0, 1, 2, \ldots$ ⟵ indices differ by integer

$$y_1 = x^{s_1} \text{ (analytic function)}$$

$$y_2 = y_1(x)\log(x) + x^{s_2} \text{ (analytic fcn)}$$

③ Other wise $P(x)$, and $Q(x)$ have stronger singularities at x_0 and x_0 is known as an irregular singular point, and is often associated with an essential singularit

Defined a Reguar Point x_0

- Where P, Q analytic at x_0
- Just try a taylor series $y = \sum_n a_n (x - x_0)^n$ and iteratively solve for the coefficients.

Ex $\left[-\frac{d^2}{dx^2} + k^2 \right] y$ This case is easy

- So $y = \sum_n a_n x^n$ $y'' = \sum_n n(n-1) a_n x^{n-2}$

$$= \sum_n (n+2)(n+1) a_{n+2} x^n$$

- Find

$$-(n+2)(n+1) a_{n+2} + k^2 a_n = 0 \Rightarrow a_{n+2} = \frac{k^2 a_n}{(n+2)(n+1)}$$

- For $a_0 = 1$ $a_1 = 0$ $y(x)\Big|_{x=0} = 1$ $y'(x)\Big|_{x=0} = 0$

$$y_1 = 1 + \frac{k^2 x^2}{1 \cdot 2} + \frac{k^4 x^4}{1 \cdot 2 \cdot 3 \cdot 4} = \cosh kx$$

- For $a_0 = 0$ $a_1 = k$ or $y(x)\Big|_{x=0} = 0$ $y'(x)\Big|_{x_0} = k$

$$y_2 = kx + \frac{k^3 x^3}{2 \cdot 3} + \frac{x^5 \cdot k^5}{2 \cdot 3 \cdot 4 \cdot 5} + \dots = \sinh kx$$

$y = C_1 \cosh kx + C_2 \sinh kx \leftarrow$ general solution

For a Regular Singular Point x_0

$$\left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\right] y = 0$$

- Where near x_0 $P(x_0)$ and $Q(x)$ behave as;

$$P(x) = \frac{p_{-1}}{x - x_0} + p_0 + \ldots\ldots$$

$$Q(x) = \frac{q_{-2}}{(x-x_0)^2} + \frac{q_{-1}}{(x-x_0)} + \ldots\ldots$$

- Take a specific example: Bessel Eqn

$$\left[x\frac{d}{dx}x\frac{d}{dx} + x^2 - \nu^2\right] y(x) = 0$$

Near $x = 0$ can drop this term

$$\left[x\frac{d}{dx}x\frac{d}{dx} - \nu^2\right] y^{(0)} = 0$$

← This is an Equidemensial Diffeq

- Substitute $y = x^s$ find:

$$s^2 - \nu^2 = 0 \Rightarrow s = \pm\nu$$ and thus

$$y^{(0)} = \underbrace{C_1 x^{\nu}}_{\text{regular sol}} + \underbrace{C_2 \frac{1}{x^{\nu}}}_{\text{irregular sol}}$$

← Leading Solution

- Now we can systematically include corrections in powers of x

$$y = y^{(0)} + y^{(1)} + y^{(2)} + \ldots$$

Then

$$\left[x\frac{d}{dx}\,x\frac{d}{dx} - \nu^2\right]y = x^2 y$$

correction

0th

$$\left[x\frac{d}{dx}\,x\frac{d}{dx} - \nu^2\right]y^{(0)} = 0$$

1st

$$\left[x\frac{d}{dx}\,x\frac{d}{dx} - \nu^2\right]y^{(1)} = 0$$

2nd

$$\left[x\frac{d}{dx}\,x\frac{d}{dx} - \nu^2\right]y^{(2)} = -x^2\,y^{(0)}$$

3rd

$$\left[x\frac{d}{dx}\,x\frac{d}{dx} - \nu^2\right]y^{(3)} = 0$$

4th

$$\left[x\frac{d}{d}\,x\frac{d}{dx} - \nu^2\right]y^{(4)} = -x^2\,y^{(2)}$$

Solve order by order

- $y^{(0)} = C_1 x^{\nu}$ ⟵ 0th
- $y^{(1)} = 0$ ⟵ 1st and same for all $y^{(3)} = y^{(5)} = \dots = 0$
- Notice that for $y^{(2)}$ the source is

$$x^2 y^{(0)} = C_1 x^{\nu+2}$$

- So make an ansatz for a particular solution:

$$y_p^{(2)} = \underbrace{K_2 x^{\nu+2}}_{\text{particular}} + \underbrace{A_1 x^{\nu} + A_2 / x^{\nu}}_{\text{homogeneous}}$$ ⟵ we can set these to zero

Then the diff eq reads

$$\left[(\nu+2)(\nu+2) - \nu^2\right] K_2 x^{\nu+2} = -x^{2+\nu} C_1$$

So

2nd $$y_p^{(2)} = \frac{C_1 x^{2+\nu}}{(\nu+2)(\nu+2) - \nu^2} = C_1 x^{\nu} \left[\frac{-x^2}{4 \cdot (\nu+1)}\right]$$

And

$$y = C x^{\nu} \left[1 - \frac{(x/2)^2}{1 \cdot (\nu+1)} + \dots \right]$$

In general keep going, find 4th etc:

$$y^{(2n+2)} = -x^2 \frac{y^{(2n)}}{(\nu+2n)(\nu+2n) - \nu^2} = y^{2n} \left[\frac{-x^2}{4 \cdot n(\nu+1)}\right]$$

Find

$$y = C x^{\nu} \left[1 - \frac{(x/2)^2}{1\cdot(\nu+1)} + \frac{x^4}{1\cdot 2\cdot(\nu+1)(\nu+2)} + \cdots \right]$$

new constant $(\Gamma(\nu+1) = \bar{C}$

divide series by $\Gamma(\nu+1)$

$$y = \bar{C} x^{\nu} \sum_n \frac{(x/2)^{2n} (-1)^n}{n!\, \Gamma(\nu+n+1)} = \bar{C}_1 J_\nu(x)$$

- For $\nu = +3/2$ and $\nu = -3/2$ a plot is shown below

- We used $\Gamma(z+1) = z\Gamma(z)$, so

$$\Gamma(\nu+1)(\nu+1)(\nu+2)(\nu+3) = \Gamma(\nu+3+1)$$

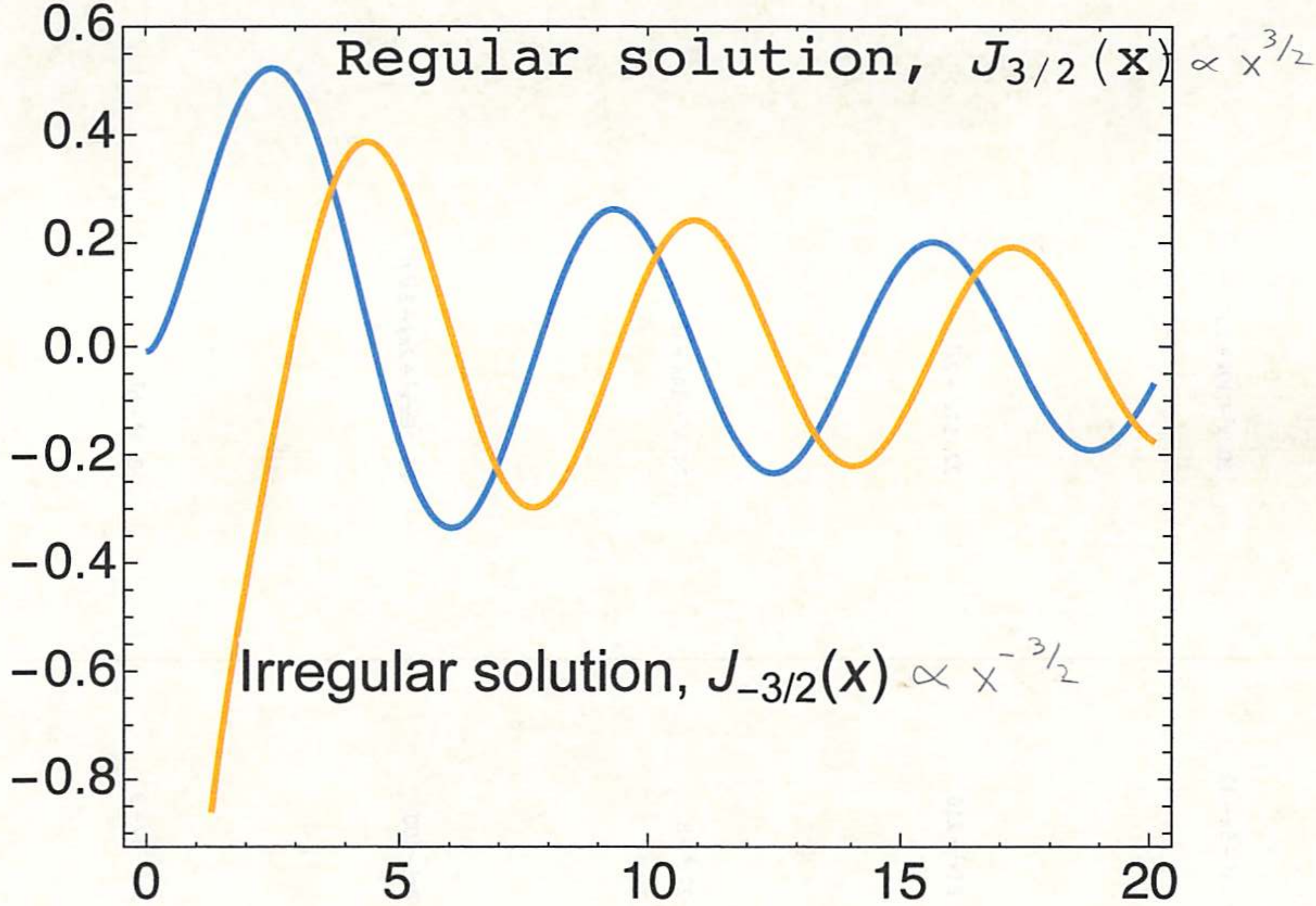
- In general the perturbation expansion yields

$$y = A_1 x^{S_1} \text{ (analytic function)}$$
$$+ A_2 x^{S_2} \text{ (analytic function)}$$

- When $S_1 = S_2$ or when they differ by an integer the perturbative expansion can be more complicated and involve log's e.g.

$$y = A_1 x^{S_1} \text{ (reg function)} + A_1 x^{S_1} \log x \text{ (reg function)}$$

We will work out these subtle case later.


Regular solution, $J_{3/2}(x) \propto x^{3/2}$
Irregular solution, $J_{-3/2}(x) \propto x^{-3/2}$
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
0
5
10
15
20

Behavior as $z \to \infty$

$$\left[\frac{d^2}{dz^2} + \mathbb{P}(z)\frac{d}{dz} + Q(z)\right] y = 0$$

- We say that $y(z)$ and the DEQ are regular / singular as $z \to \infty$, by looking at y as a function of $w = \frac{1}{z}$. Define (see picture)

$$\bar{y}(w) = y(z) \qquad \text{with } w = 1/z$$

Then

$$\frac{d\bar{y}}{dw} = \frac{dy}{dz}\frac{dz}{dw} \longrightarrow -w^2\frac{d\bar{y}}{dw} = \frac{dy}{dz}$$

$z = 1/w$

Also note $-w\frac{d}{dw} = z\frac{d}{dz}$. The transformed

- DEQ is:

$$\left[\frac{d^2}{dw^2} + \bar{\mathbb{P}}(w)\frac{d}{dw} + \bar{Q}(w)\right]\bar{y} = 0 \qquad \bar{P} = \frac{2}{w} - \frac{1}{w^2}P\left(\frac{1}{w}\right)$$

$$\bar{Q} = \frac{1}{w^4}Q(w)$$

What is the nature of the singularity (if any) at $w = 0$?