D_3 Matrix Representation Summary

	r_0	r_1	r_2	s_0	s_1	s_2
Identity	1	1	1	1	1	1
Alternate	1	1	1	-1	-1	-1
Matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$
Regular	$\mathbb{1}_{6\times 6}$	see multiplacation table				

- Here $\mathbb{1}_{6\times 6}$ is the 6×6 identity matrix. Some matrix representatives of the regular representation are

$$D(r_1) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad D(s_0) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

D_3 Matrix Irreducable Representation Summary and Orthogonality

	r_0	r_1	r_2	s_0	s_1	s_2
Identity	1	1	1	1	1	1
Alternate	1	1	1	-1	-1	-1
Matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$

The characters are

	$\chi(r_0)$	$\chi(r_1)$	$\chi(r_2)$	$\chi(s_0)$	$\chi(s_1)$	$\chi(s_2)$
Identity $(\mu) = 1$	$1 = n_1$ (dimension of rep)	1	1	1	1	1
Alternate $(\mu) = 2$	$1 = n_2$ (dimension of rep)	1	1	-1	-1	-1
Matrix $(\mu) = 3$	$2 = n_3$ (dimension of rep)	-1	-1	0	0	0

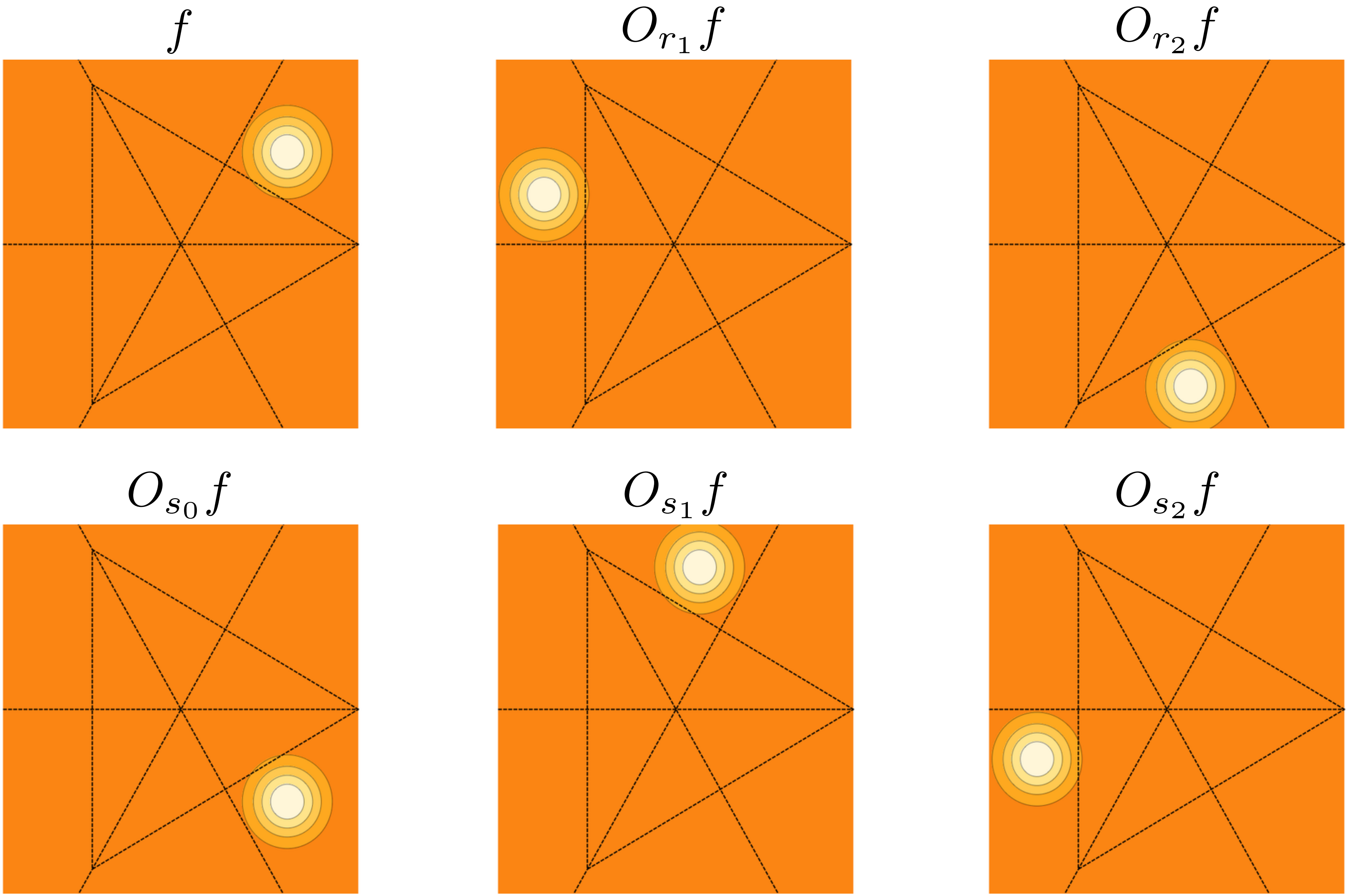
The character orthogonality of matrix and alternate rep reads

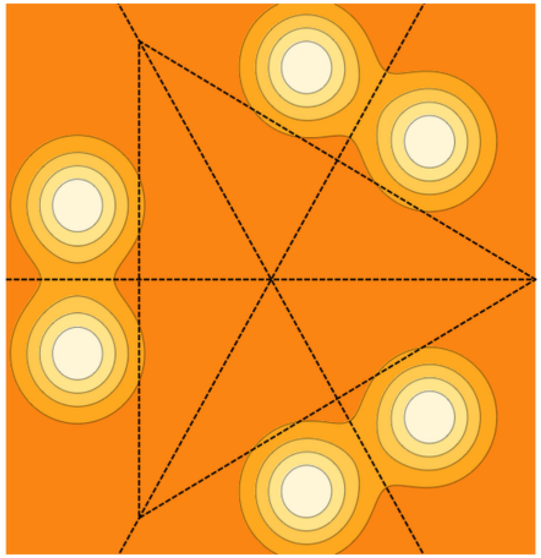
$$\sum_g \chi^{(2)}(g)\chi^{(3)*}(g) = 1 \cdot 2 + 1 \cdot (-1) + 1 \cdot (-1) = 0$$

The dimensions of reps add up to order of group:

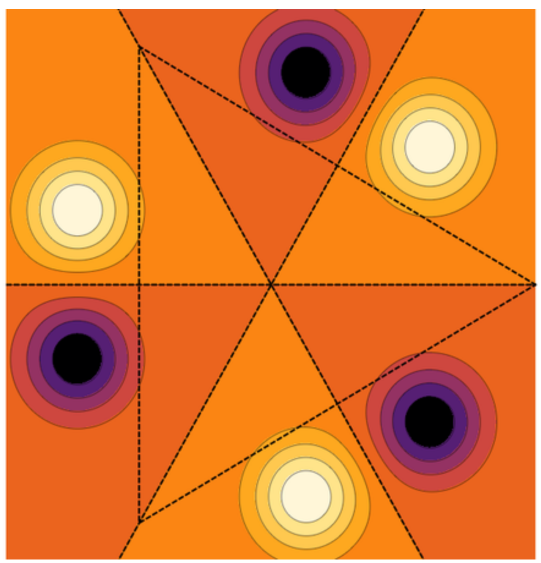
$$\sum_\mu n_\mu^2 = 1^2 + 1^2 + 2^2 = 6 = n_G = \text{order of group}$$

The vector space = linear span of six functions

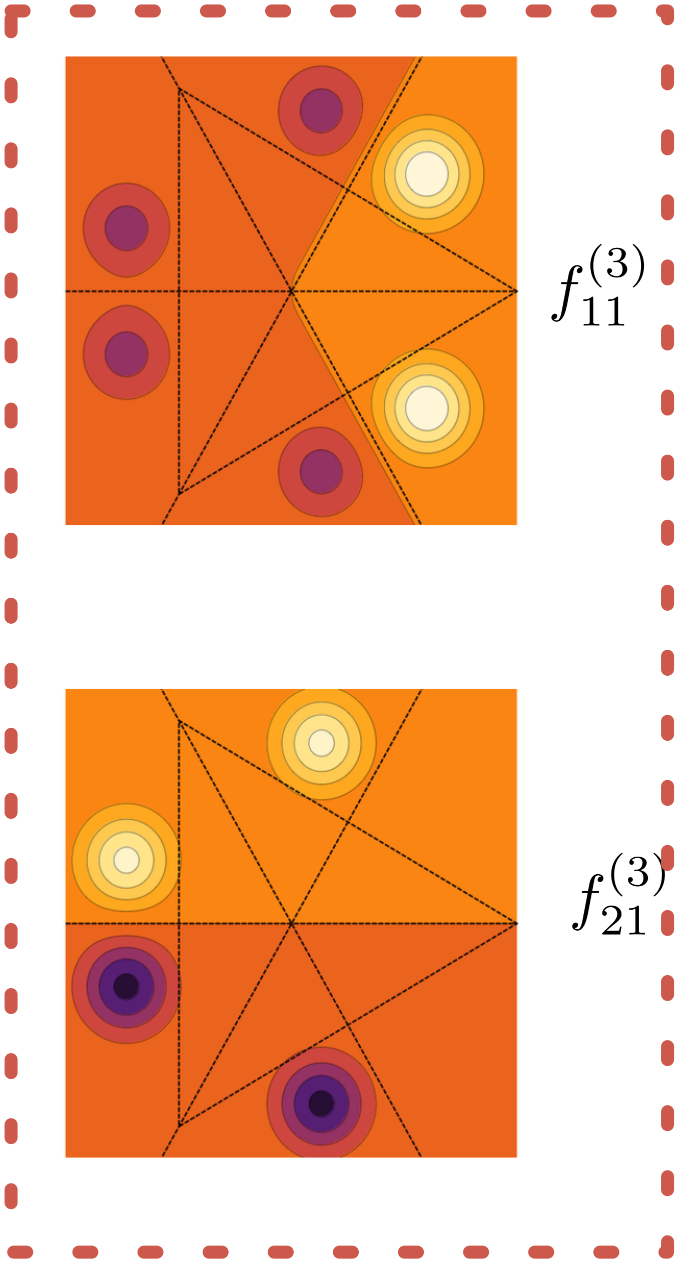




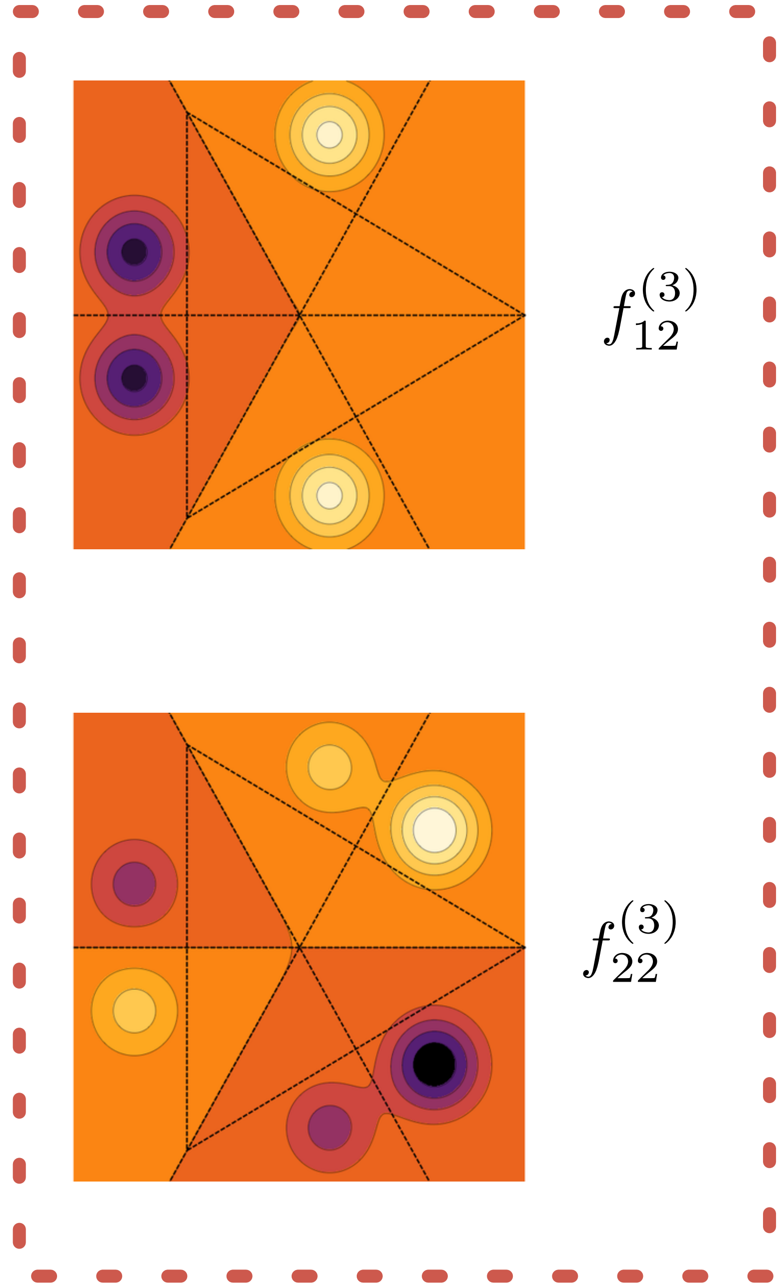
$$f_S(x) \equiv f_{11}^{(1)}(x)$$



$$f_A(x) \equiv f_{11}^{(2)}$$



Partners in an irreducible rep

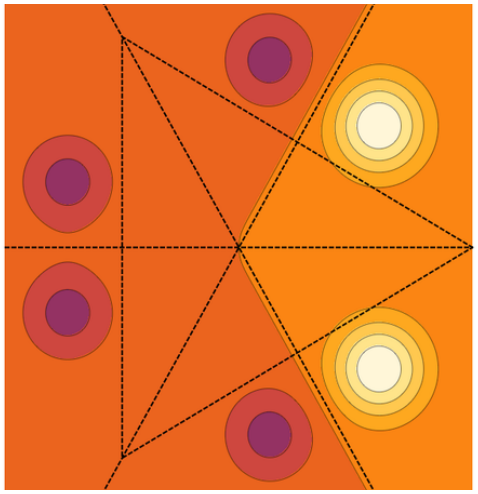


Partners in an irreducible rep

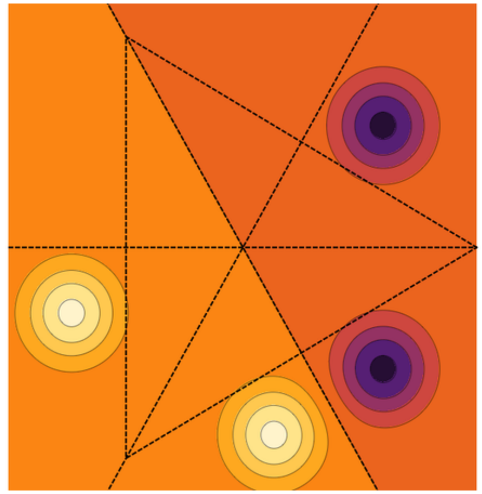
Partners in the same irrep are mixed by the group operations

(but different irreps do not mix)

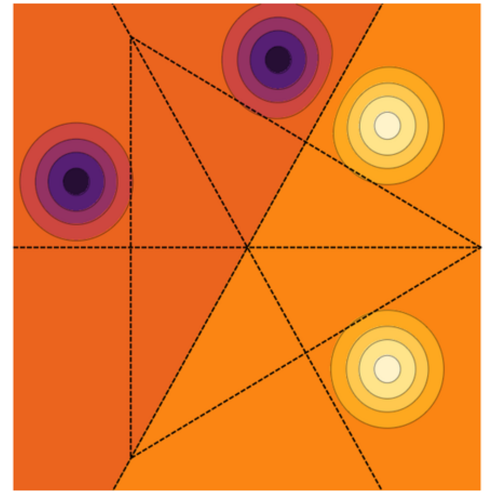
$f_{11}^{(3)}$



= -1/2

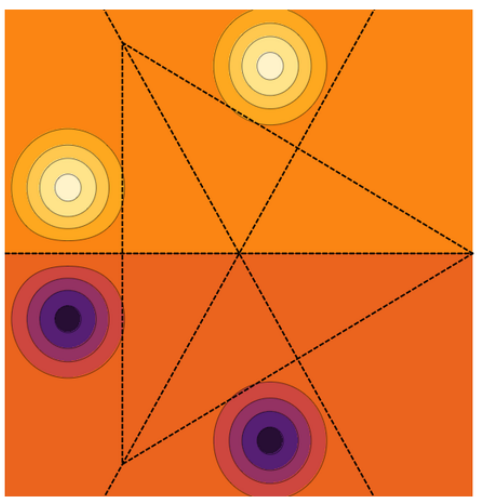


+ 1/2

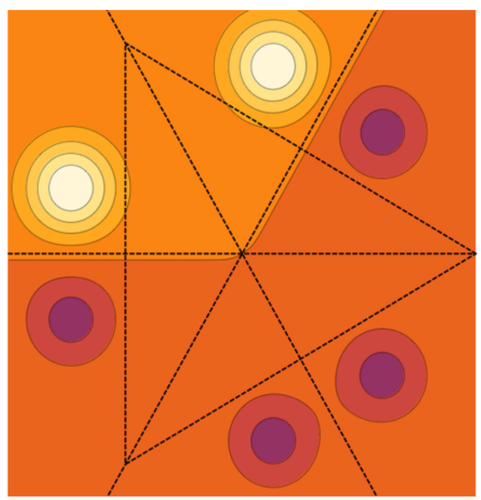


$$f_{11}^{(3)} = \frac{1}{2}(-O_{r_1} + O_{r_2})f_{21}^{(3)}$$

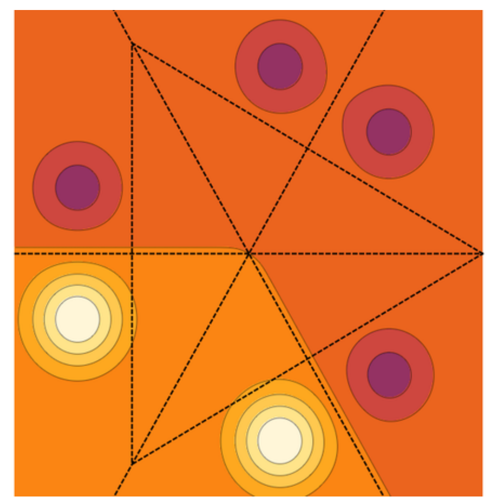
$f_{21}^{(3)}$



= 1/2

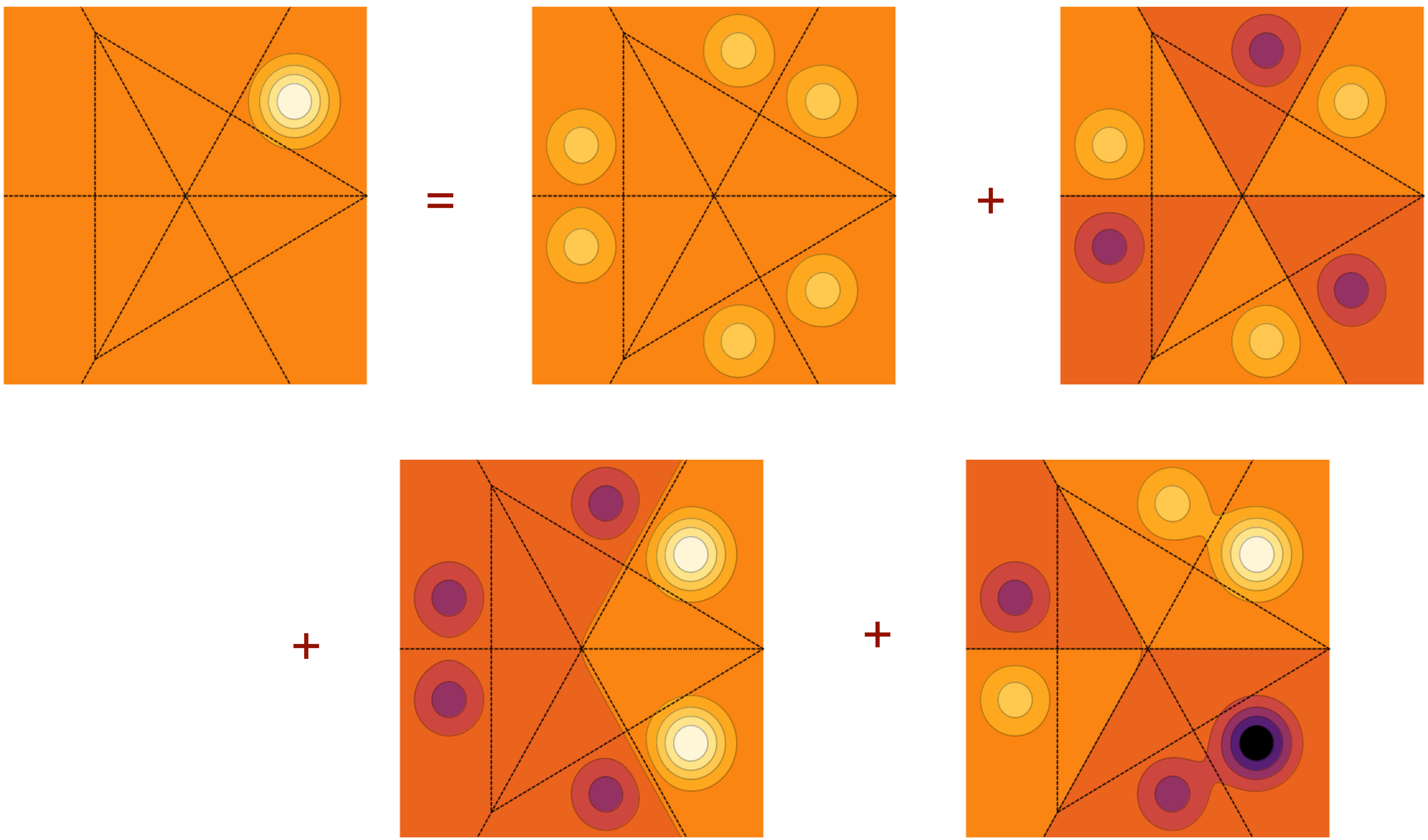


- 1/2



$$f_{21}^{(3)} = \frac{1}{2}(O_{r_1} - O_{r_2})f_{11}^{(3)}$$

Projection theorem portrayed graphically



$$f = \sum_{\mu,a} f_{aa}^{(\mu)} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} + f_{22}^{(3)}$$

D_3 Character Table and Orthogonality

number in class	$\chi(C_1)$ $n_I{=}1$	$\chi(C_2)$ $n_I{=}2$	$\chi(C_3)$ $n_I{=}3$
Identity $(\mu) = 1$	1	1	1
Alternate $(\mu) = 2$	1	1	-1
Matrix $(\mu) = 3$	2	-1	0

Then the sum over characters squared

$$\sum_I |\chi^{(\mu)}(C_I)|^2 n_I = n_G \qquad e.g. \qquad 1^2 + 1^2 \cdot 2 + 1^2 \times 3 = 6$$

But the second and third row are arthogonal

$$\sum_I n_I \, \chi^{(\mu)}(C_I)(\chi^{(\nu)}(C_I))^* = n_G \delta_{\mu\nu} \qquad e.g. \qquad 1 \cdot 2 + 1 \cdot (-1) \times 2 = 0$$

The columns are orthogonal

$$\sum_{\mu} \chi^{(\mu)}(C_I)(\chi^{(\mu)}(C_J))^* = \frac{n_G}{n_I}\delta_{IJ} \qquad e.g. \qquad 1\cdot 1 + 1\cdot 1 + 2\cdot(-1) = 0$$

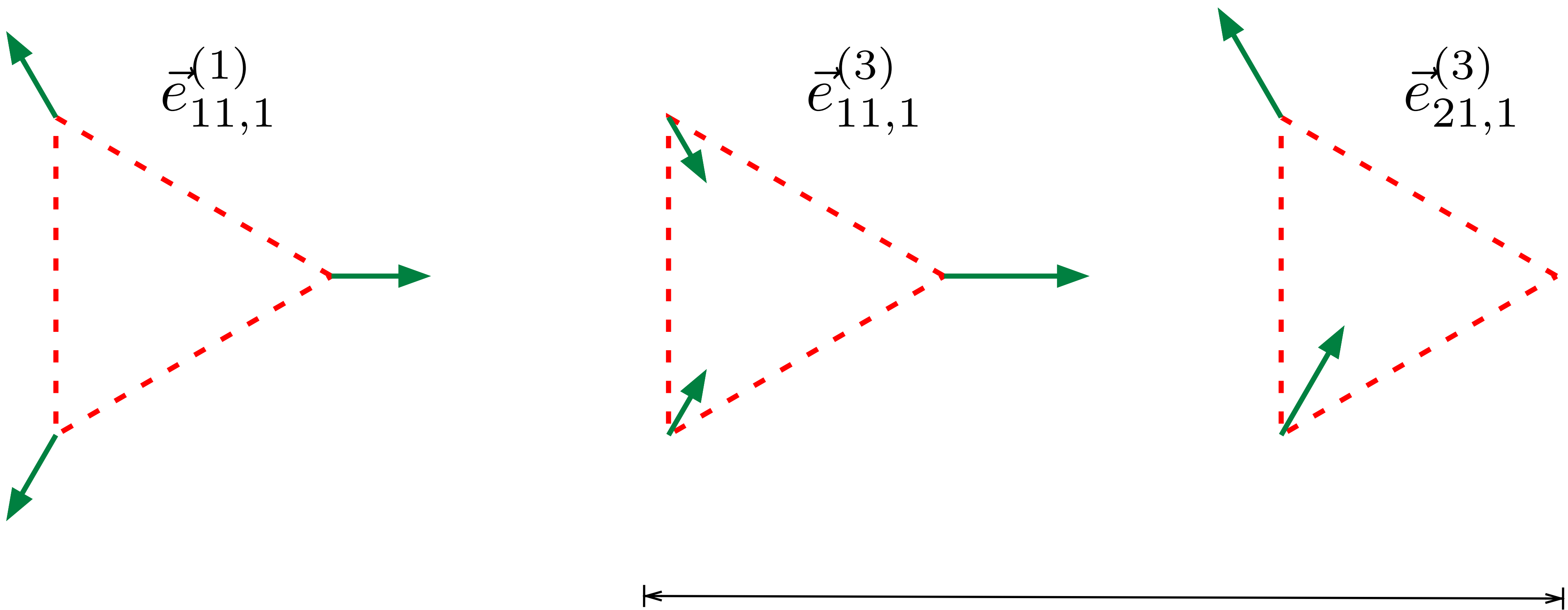
$$O_{\mathbb{1}} = \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} \quad O_{r_1} = \begin{pmatrix} 0 & 0 & D \\ D & 0 & 0 \\ 0 & D & 0 \end{pmatrix} \quad O_{r_2} = \begin{pmatrix} 0 & D & 0 \\ 0 & 0 & D \\ D & 0 & 0 \end{pmatrix}$$

$$O_{s_0} = \begin{pmatrix} 0 & 0 & D \\ D & 0 & 0 \\ 0 & D & 0 \end{pmatrix} \quad O_{s_1} = \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & D \\ 0 & D & 0 \end{pmatrix} \quad O_{s_2} = \begin{pmatrix} 0 & 0 & D \\ 0 & D & 0 \\ D & 0 & 0 \end{pmatrix}$$

where the D in O_{s_2} (for example) is short for $D^{(3)}(s_2)$, and analogously for the other operators.

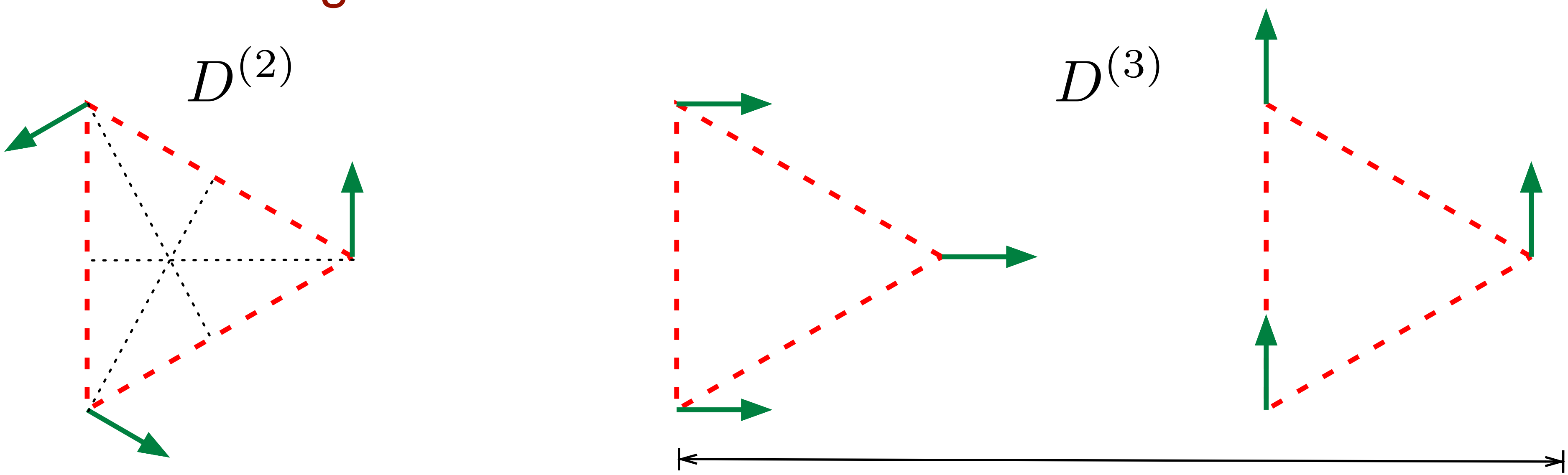
	r_0	r_1	r_2	s_0	s_1	s_2
$D^{(3)}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix}$

Three vectors in the space

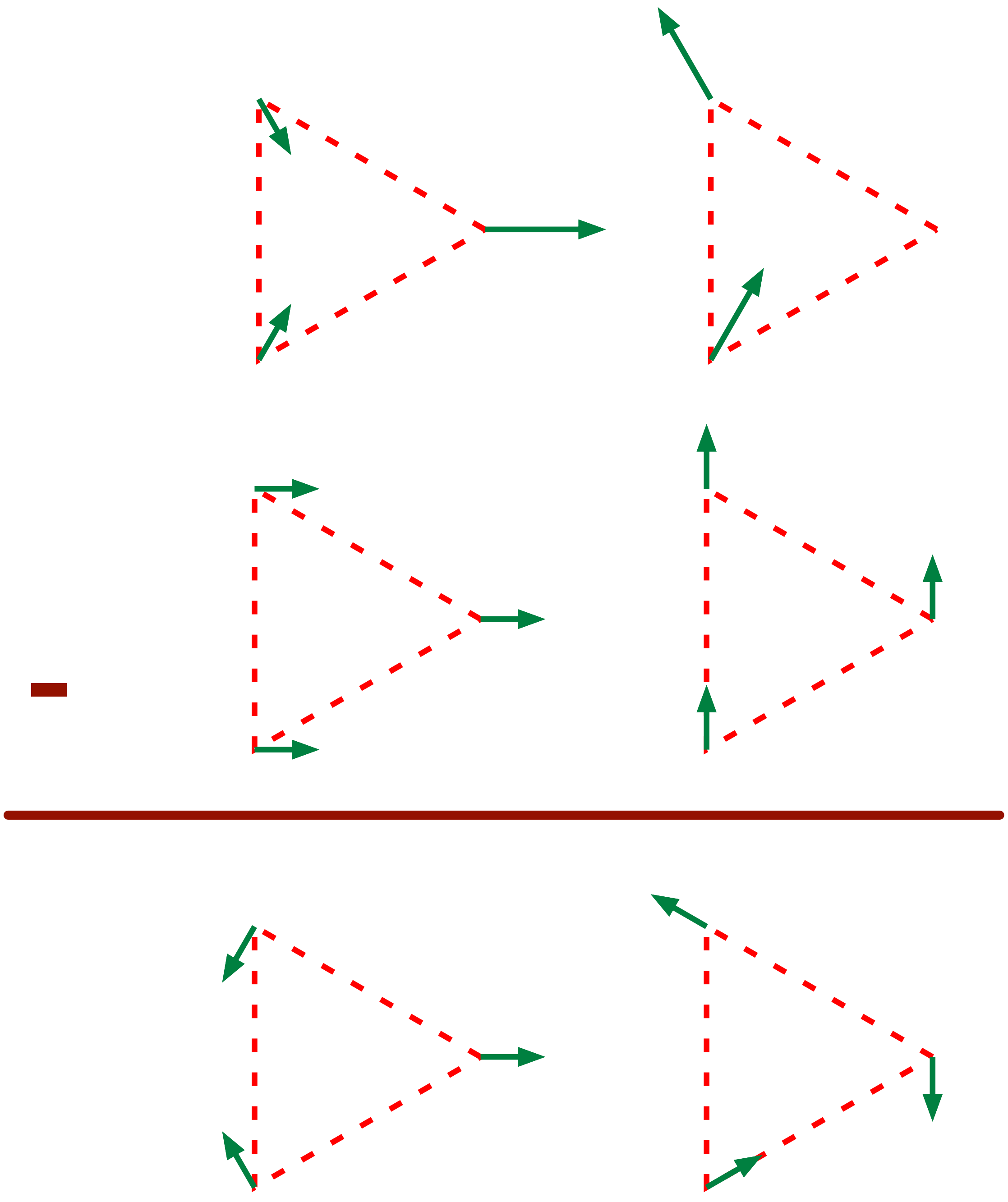


Partners in the $D^{(3)}$ irreducible rep

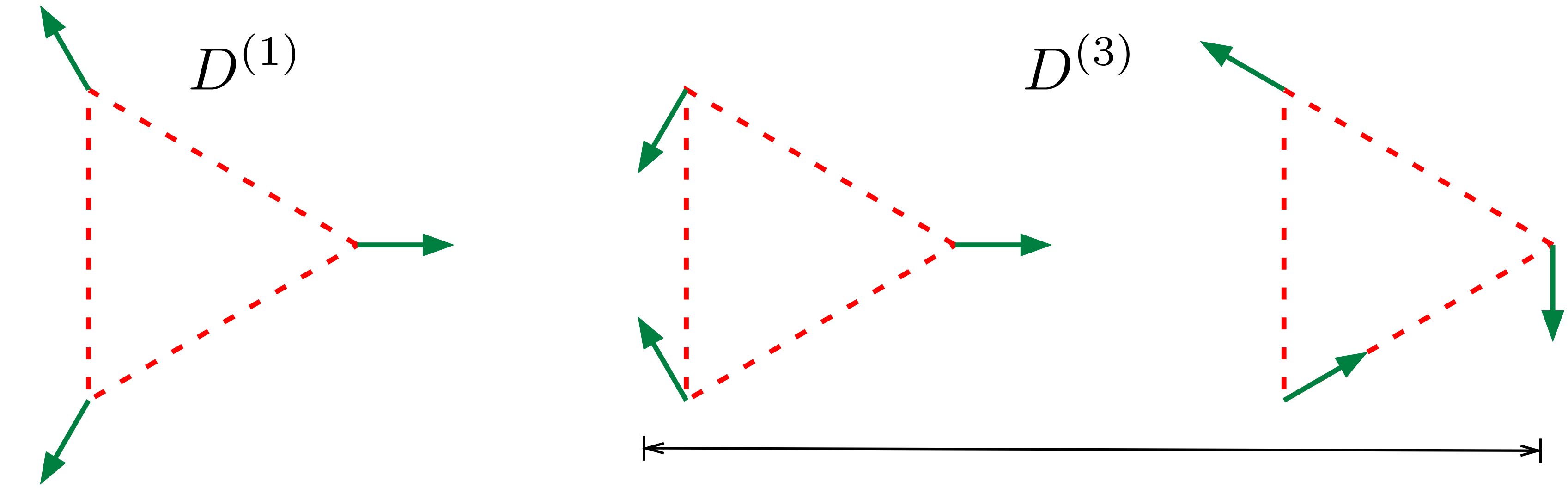
Zero Mode Eigenvectors



Subtracting Center of Mass Motion or Zero Modes



Vibrational Modes



Zero Modes