Levi-Cevita Practice

$$A_k = -\frac{1}{2} \varepsilon_{klm} r^l B^m$$

Then

$$(\nabla \times A)^i = \varepsilon^{ijk} \partial_j \frac{-1}{2} \varepsilon_{klm} r^l B^m$$

$$= \varepsilon^{ijk} \frac{-1}{2} \varepsilon_{kjm} B^m$$

$$\varepsilon^{ijk} \varepsilon_{jmk} = \delta^i_j \delta^j_m - \delta^i_m \delta^j_j$$

$$= -2\delta^i_m$$

So

$$(\nabla \times A)^i = B^i$$

Picture:

$$\oint \vec{A} \cdot d\vec{l} = \int B \cdot d\vec{S}$$

$$A^\phi \, 2\pi r = B_z \pi r^2$$

$$A_\phi = \frac{1}{2} B_z r$$

flux of B

causes circulation of $\vec{A}$

b) Determinants

(1) $$A^{i_1}{}_{j_1} A^{i_2}{}_{j_2} A^{i_3}{}_{j_3} \, \varepsilon^{j_1 j_2 j_3} = \det A \, \varepsilon^{i_1 i_2 i_3}$$

(2) $$A_{j_1}{}^{i_1} A_{j_2}{}^{i_2} A_{j_3}{}^{i_3} \, \varepsilon^{j_1 j_2 j_3} = \det A^T \, \varepsilon^{i_1 i_2 i_3}$$

Multiply both sides of (2) by $\varepsilon_{i_1 i_2 i_3}$ and sum

$$\left[A_{j_1}{}^{i_1} A_{j_2}{}^{i_2} A_{j_3}{}^{i_3} \, \varepsilon_{i_1 i_2 i_3} \right] \varepsilon^{j_1 j_2 j_3} = \det A^T \; 3!$$

Look at the term is square brackets, and compare to Eq (1). They are the same. The second indices are the ones contracted with ε-tensor in both cases. Thus

$$(\det A) \underbrace{\varepsilon_{j_1 j_2 j_3} \varepsilon^{j_1 j_2 j_3}}_{3!} = \det A^T \; 3!$$

Similarly

$$\text{LHS} = (A^{i_1}{}_{l_1} B^{l_1}{}_{j_1})(A^{i_2}{}_{l_2} B^{l_2}{}_{j_2})(A^{i_3}{}_{l_3} B^{l_3}{}_{j_3}) \, \varepsilon^{j_1 j_2 j_3}$$

$$= (\det AB) \, \varepsilon^{i_1 i_2 i_3}$$

Then

$$\text{LHS} = A^{i_1}{}_{l_1} A^{i_2}{}_{l_2} A^{i_3}{}_{l_3} \left[B^{l_1}{}_{j_1} B^{l_2}{}_{j_2} B^{l_3}{}_{j_3} \, \varepsilon^{j_1 j_2 j_3} \right]$$

same!

$$= A^{i_1}{}_{l_1} A^{i_2}{}_{l_2} A^{i_3}{}_{l_3} \, (\det B) \, \varepsilon^{l_1 l_2 l_3} = (\det A)(\det B) \, \varepsilon^{i_1 i_2 i_3}$$

$$\nabla \times \frac{\vec{r}}{r^2} = 0$$

$$\left(\nabla \times \frac{\vec{r}}{r^2}\right)^i = \varepsilon^{ijk}\, \partial_j \frac{r_k}{r^2}$$

$$= \varepsilon^{ijk}\left[\frac{\delta_{jk}}{r^2} - \frac{r_k}{r^4}\partial_j (r^\ell r_\ell)\right]$$

$$= \frac{\varepsilon^{ijk}}{r^2}\left[\delta_{jk} - 2\frac{r_j r_k}{r^2}\right] = 0$$

The last step follows because ε^{ijk} is antisymmetric in j, k, while the tensor in square brackets is symmetric

d) Magnetic Field of a dipole:

$$B^i = (\nabla \times A)^i = \varepsilon^{ijk}\, \partial_j \left[\varepsilon_{k\ell o} \frac{m^\ell r^o}{4\pi r^3}\right]$$

$$= \frac{\varepsilon^{ijk}}{4\pi} \varepsilon_{\ell o k}\left[\frac{m^\ell \delta^o_j}{r^3} - 3\frac{m^\ell r^o}{r^5} r_j\right]$$

We used $\partial_j r = n_j = \frac{r_j}{r}$ so $\partial_j r^\alpha = \alpha r^{\alpha - 1} n_j$, e.g.:

$$\partial_j \frac{1}{r^3} = \partial_j \frac{1}{(r^\ell r_\ell)^{3/2}} = \frac{-3}{2}\frac{1}{(r^\ell r_\ell)^{5/2}}(r^\ell \delta_{\ell j} + \delta^\ell_j r_\ell)$$

Then

$$B^i = (\delta^i_l \delta^j_0 - \delta^i_0 \delta^j_l) \frac{m^l}{4\pi r^3} [\delta^0_j - 3 n^0 n_j]$$

Then contracting the l-index and simplifying:

$$= \frac{1}{4\pi r^3} \left[(m^i \delta^j_0 - \delta^i_0 m^j)(\delta^0_j - 3 n^0 n_j) \right]$$

$$[\,] = \underbrace{3m^i}_{①} \underbrace{- m^i}_{②} \underbrace{- 3m^i (\vec{n}\cdot\vec{n})}_{④} + \underbrace{3(\vec{m}\cdot\vec{m})\, n^i}_{③}$$

Using $\vec{n}\cdot\vec{n} = 1$ we find the expected result

$$B^i = \frac{3(\vec{n}\cdot\vec{m})\, n^i - m^i}{4\pi r^3}$$

Simple Application of Helmholtz

(1) $\nabla \cdot E = \rho$

(2) $\nabla \times \vec{B} = \vec{j}/c + \frac{1}{c} \partial_t \vec{E}$

Take divergence of (2), use $\nabla \cdot (\nabla \times B) = 0$ yielding

$$0 = \frac{1}{c} \nabla \cdot j + \frac{1}{c} \partial_t \vec{\nabla} \cdot \vec{E}$$

So with Eq (1) and multiplying by c give

$$\partial_t \rho + \nabla \cdot j = 0$$

The multipole expansion

a) Writing

$$\frac{1}{|\vec{r}-\vec{x}|} = \frac{1}{r}\left(1 - \frac{2r_i x^i}{r^2} + \frac{x^2}{r^2}\right)^{-1/2}$$

Using $(1+z)^{-1/2} = 1 - \frac{1}{2}z + \frac{3}{8}z^2 + \ldots$ find

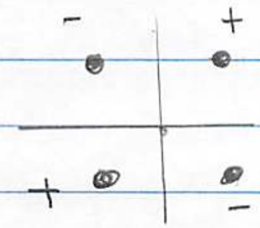
$$= \frac{1}{r}\left(1 - \frac{r_i x^i}{r^2} - \frac{1}{2}\frac{x^2}{r^2} + \frac{3}{8}\left(\frac{2r_i x^i}{r^2}\right)^2 + \ldots\right)$$

$$= \frac{1}{r}\left(1 - \frac{r_i x^i}{r^2} + \frac{1}{2}\frac{r^i r^j}{r^4}(3x_i x_j - x^2\delta_{ij}\right)$$

b) Thus integrating over $\vec{x}$

$$\phi(r) = \frac{1}{4\pi}\left[\frac{q_{tot}}{r} + \frac{p_i r^i}{r^3} + \frac{1}{2}\frac{Q^{ij} r_i r_j}{r^4} + \ldots\right]$$

b)



$$Q^{ij} = \sum_a q_a \left(3x_a^i x_a^j - x_a^2 \delta^{ij}\right)$$

$$\vec{X}_a = (\pm a, \pm a, 0)$$

$$x_a^2 = 2a^2$$

① $$Q^{xx} = q\,(3\cdot a^2 - 2a^2) - q\,(3a^2 - 2a^2) + q\,(3a^2 - 2a^2) - q\,(3a^2 - 2a^2) = 0$$

② $Q^{yy} = 0$ by symmetry

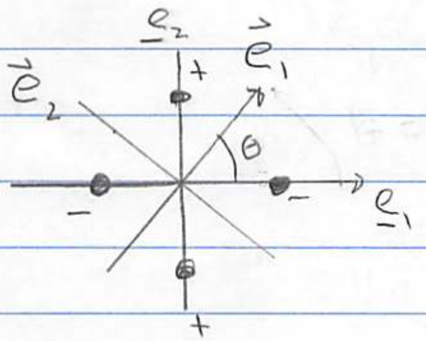
③ $$Q^{xy} = q\,(3a^2) - q\,(3(-a)(a)) + q\,(3(-a)(-a)) - q\,(3a(-a))$$

④ $$Q^{xy} = 12\,q\,a^2$$

Summarizing

(i) $\mathcal{Q}^{ij} = 12qa^2 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Now we need to construct the appropriate coordinate system. At a time t, the angle $\theta = \omega t$ as drawn below. We constructed the appropriate rotation matrix on the first day



$$(\mathcal{R})^i{}_j = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Then working in the 2×2 block of the matrix

$$\underline{Q}^{ij} = (\mathcal{R})^i{}_m (\mathcal{R})^j{}_l \, Q^{ml}$$

$$= (\mathcal{R})^i{}_m \, Q^{ml} (\mathcal{R}^T)_l{}^j$$

$$\underline{Q}^{ij} = (\mathcal{R})^i{}_m \, Q^{ml} (\mathcal{R}^{-1})_l{}^j$$

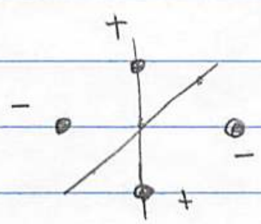
$$\left(\underline{Q}^{ij} \right) = \left(\mathcal{R} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\mathcal{R}^T \right) \cdot 12qa^2$$

$$\left(\underline{Q}^{ij} \right) = \begin{pmatrix} -\sin(2\theta) & \cos 2\theta \\ \cos(2\theta) & \sin 2\theta \end{pmatrix}$$

Putting in a 3x3 form

$$\mathcal{Q}^{ij} = \left(\begin{array}{cc|c} -\sin(2\omega t) & \cos(2\omega t) & 0 \\ \cos(2\omega t) & \sin(2\omega t) & 0 \\ 0 & 0 & 1 \end{array} \right) 12 q a^2$$

If $\omega t = \pi/4$ then $\mathcal{Q}^{xx}$ is determined by the negative charges



$$\mathcal{Q}^{xx} = \sum_a q_a \left(3 x_a^x x_a^x - (2a^2) \delta^{11} \right)$$

$$= -q \left(3 \; 2a^2 \right) \times 2 = -12 q a^2$$

TBP Like mad!

a) $$\int_V \vec{F} \times \vec{G} = \int_V \varepsilon^{ijk} \partial_j \phi \, G_k$$

$$= \int_V \varepsilon^{ijk} \phi (-\partial_j G_k) = \int_V d^3x \, \phi \, \vec{\nabla} \times \vec{G} = 0$$

b) This follows like this

$$\int_V d^3x \, j^l = \int_V d^3x \, j^i \frac{\partial x^l}{\partial x^i}$$

$$= \int d^3x - \partial_i j^i \, x^l = 0$$

Since $\nabla \cdot \vec{j} = \partial_i j^i = 0$

c) $$\int dS_i = \int d\vec{S} \cdot \vec{e}_i = \int d\vec{S} \cdot \nabla \times -\frac{1}{2} \vec{r} \times \vec{e}_i$$

$$= \int d\vec{l}_i \cdot \left(-\frac{1}{2} \vec{r} \times \vec{e}_i\right)$$

Using $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

$$\int dS_i = \int \vec{e}_i \cdot \left(d\vec{l} \times -\frac{1}{2} \vec{r}\right) = \frac{1}{2} \int e_i \cdot (\vec{r} \times d\vec{l})$$

or

$$\int dS_i = \frac{1}{2}\oint (\vec{r} \times d\vec{l})_i$$

d) $$\oint dS_i = \oint d\vec{S} \cdot \vec{e}_i$$

$$= \int_V d^3x \; \nabla \cdot \vec{e}_i = 0$$

But $\vec{e}_i$ is a constant vectror, Its divergence is zero.

Similarly

$$\oint d\vec{S} \cdot \vec{r} = \int_V d^3x \; \vec{\nabla} \cdot \vec{r} = \int_V d^3x \; \overbrace{\partial_i x^i}^{= \delta^i_i = 3}$$

$$= 3V$$