Volumes And Dual bases

a) Let $\varepsilon_{abc} = \sqrt{g}\,[abc]$ then raising indices

$$\varepsilon^{abc} = \sqrt{g}\, g^{aa'} g^{bb'} g^{cc'} [a'b'c']$$

Now consider g_{ab} a matrix $(g)_{ab}$. Its inverse is $(g^{-1})^{ab}$. We know that

$$(g^{-1})^{ab} = g^{ab}$$

i.e. $(g)(g^{-1}) = \mathbb{1}$ or $(g)_{ab}(g^{-1})^{bc} = g_{ab}g^{bc} = \delta^c_a$

Then

$$g^{aa'} g^{bb'} g^{cc'} [a'b'c'] = [abc]\det(g^{-1})$$

Thus since $\det(g^{-1}) = 1/\det(g)$ we have

$$\varepsilon^{abc} = \sqrt{\det(g)}\;\frac{1}{(\det g)}\;[abc]$$

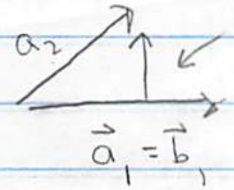
$\sqrt{g} \equiv \sqrt{\det g_{ab}}$

$$= \frac{1}{\sqrt{\det(g)}}[abc]$$

$$= \frac{1}{\sqrt{g}}[abc]$$

Dual Basis

b) We have the picture:

i) • 

$\vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1} \vec{b}_1$ = the component of $\vec{a}_2 \perp \vec{b}_1 = \vec{b}_2$

• Clearly $\vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1} \vec{b}_1$ is orthogonal to $\vec{b}_1$

since

$$(\vec{a}_2 \cdot \vec{b}_1) - \frac{(\vec{a}_2 \cdot \vec{b}_1)}{\cancel{(\vec{b}_1 \cdot \vec{b}_1)}} \cancel{(\vec{b}_1 \cdot \vec{b}_1)} = 0$$

ii) $\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \det(\vec{a}_1, \vec{a}_2, \vec{a}_3)$

$$= \det(\vec{b}_1, \vec{a}_2, \vec{a}_3)$$

Now for $\vec{a}_2$ we may add an arbitrary amount of $\vec{b}_1$ since, $\det(\vec{b}_1, \vec{b}_1, \vec{a}_3) = 0$

$$\det(\vec{b}_1, \vec{a}_2 - c\vec{b}_1, \vec{a}_3) = \det(\vec{b}_1 \vec{a}_2 \vec{a}_3) - c \underbrace{\cancel{\det(\vec{b}_1 \vec{b}_1 \vec{a}_3)}}_{0}$$

Continuing in this way we see that

$$\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \det(\vec{b}_1\ \vec{b}_2\ \vec{b}_3)$$

Finally assembing $\vec{b}_1\ \vec{b}_2\ \vec{b}_3$ into a matrix

$$B = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

Note that $B^T B$ is diagonal

$$B^T B = \begin{pmatrix} \vec{b}_1 \cdot \vec{b}_1 & 0 & 0 \\ 0 & \vec{b}_2 \cdot \vec{b}_2 & 0 \\ 0 & 0 & \vec{b}_3 \cdot \vec{b}_3 \end{pmatrix}$$

since the $\vec{b}$'s are orthogonal. Taking the determinant is simple

$$\det(B^T B) = (\det B)^2 = (\vec{b}_1 \cdot \vec{b}_1)(\vec{b}_2 \cdot \vec{b}_2)(\vec{b}_3 \cdot \vec{b}_3)$$

and finally taking the sqrt (while noting that $\det B > 0$) gives

$$\det(\vec{a}_1 \vec{a}_2 \vec{a}_3) = \det(\vec{b}_1 \vec{b}_2 \vec{b}_3) = |\vec{b}_1| |\vec{b}_2| |\vec{b}_3|$$

c) Given $\vec{g}_1, \vec{g}_2, \vec{g}_3$ we look for $g^a = \vec{g}^1, \vec{g}^2, \vec{g}^3$

i) such that

$$\vec{g}^a \cdot \vec{g}_b = \delta^a{}_b$$

• Definition of Dual Basis

Clearly

$$\vec{g}^1 = (\vec{g}_2 \times \vec{g}_3)/\omega$$

$$\vec{g}^2 = (\vec{g}_3 \times \vec{g}_1)/\omega$$

$$\vec{g}^3 = (\vec{g}_1 \times \vec{g}_2)/\omega$$

Does the job, e.g.:

$$\vec{g}_1 \cdot \vec{g}^1 = \vec{g}_1 \cdot (\vec{g}_2 \times \vec{g}_3)/\omega = \frac{\omega}{\omega} = 1.$$

But $\vec{g}_2 \cdot \vec{g}^1$ and $\vec{g}_3 \cdot \vec{g}^1 = 0$, since

$\vec{g}_2 \times \vec{g}_3$ is orthogonal to $\vec{g}_2$ and $\vec{g}_3$

ii) Then we assemble $(\vec{g}_1, \vec{g}_2, \vec{g}_3)$ into a matrix (g):

$$\det(g) = \det(\vec{g}_1 \vec{g}_2 \vec{g}_3) = \vec{g}_1 \cdot (\vec{g}_2 \times \vec{g}_3) = \omega = \begin{vmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \\ \vdots & \vdots & \vdots \end{vmatrix}$$

While if we assemble $\vec{g}^1, \vec{g}^2, \vec{g}^3$ into rows of a matrix:

$$(g^{-1}) = \begin{pmatrix} \vec{g}^1 & \cdot & \cdot \\ \vec{g}^2 & \cdot & \cdot \\ \vec{g}^3 & \cdot & \cdot \end{pmatrix}$$

it forms the inverse of $(g) = \begin{pmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \\ \vdots & \vdots & \vdots \end{pmatrix}$. Just multiply it out to prove this, using $\vec{g}_a \cdot \vec{g}^b = \delta^a_b$.

Now

$$\det \begin{pmatrix} \vec{g}^1 & \cdot & \cdot \\ \vec{g}^2 & \cdot & \cdot \\ \vec{g}^3 & \cdot & \cdot \end{pmatrix} = \vec{g}^1 \cdot (\vec{g}^2 \times \vec{g}^3)$$

And use $\det(g^{-1}) = 1/\det g$ to conclude that

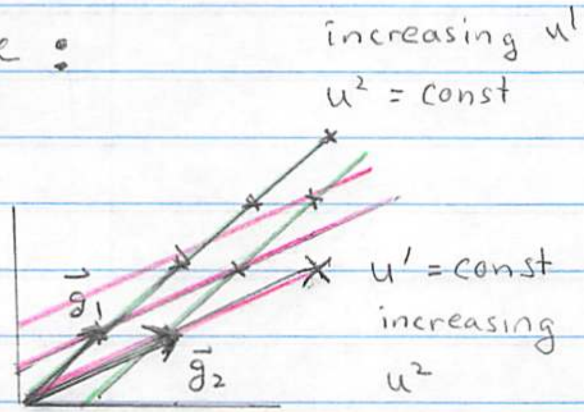
$$\vec{g}^1 \cdot (\vec{g}^2 \times \vec{g}^3) = \frac{1}{w}$$

A Dot-Product in non-orthogonal coordinates:

$$x^1 = u^1 + 2u^2$$

$$x^2 = u^2 + u^1 \qquad x^2 \equiv y$$

Then the basis vectors are:



$$\vec{g}_a = \frac{\partial x^i}{\partial u^a} \vec{e}_i$$

$$\vec{g}_1 = \hat{x} + \hat{y}$$

$$\vec{g}_2 = 2\hat{x} + \hat{y}$$

So

$$\vec{g}_1 \cdot \vec{g}_1 = 2$$

$$\vec{g}_2 \cdot \vec{g}_2 = 5 \qquad \text{and} \qquad g_{ab} = \vec{g}_a \cdot \vec{g}_b = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

$$\vec{g}_1 \cdot \vec{g}_2 = 3$$

The inverse metric is

$$(g^{-1})^{ab} \equiv g^{ab} = \frac{1}{\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$$

Then

$$\vec{u} \cdot \vec{v} = g^{ab} u_a v_b$$

$$= 5 (u_1 v_1) - 3 (u_1 v_2 + u_2 v_1) + 2 u_2 v_2$$