All Arfken problems are reproduced at the end of this assignment

Problem 1. Arfken 11.2.11.

Problem 2. Integrals

(a) Do Arfken 11.3.3

(b) The integral

$$\int_{3+4i}^{4-3i} (x^2 - iy^2)\, dz \tag{1}$$

is not defined without specifying the path. Explain on general grounds why this integeral depends on the path, while the previous excercise does not.

Problem 3. Integrals around loops

- Arfken, 11.3.7
- Evaluate

$$\oint \frac{dz}{(2z-1)(2z+1)^2(z+3)} \tag{2}$$

for the contour the unit circle with clockwise orientation

Problem 4. Trignometric integrals

(a) Learn about trignometric integrals by reading **Example 11.8.1** in Arfken and evaluate the integral

$$I(a) = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} \tag{3}$$

for real $a < 1$.

(b) Do Arfken 11.8.3.

(c) Consider $I(a)$ in Eq. (3) to be an analytic function a. Where is the nearest singularity of $I(a)$ to the origin, and why is this the expected result.

Problem 5. Some Fourier Integrals

(a) Recall that the fourier transfrom, $f(\omega) = \int dt e^{+i\omega t} f(t)$, of exponential

$$e^{-\gamma|t|} \tag{4}$$

is a Lorentzian:

$$\frac{2\gamma}{\omega^2 + \gamma^2} \tag{5}$$

Using complex analysis, show that inverse fourier transform of Eq. (5) is Eq. (4), i.e. show by contour integration

$$\int_{\infty}^{-\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{2\gamma}{\omega^2 + \gamma^2} = e^{-\gamma|t|} \tag{6}$$

(b) For ϵ small but finite, show that

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{i}{\omega + i\epsilon} = \theta(t) e^{-\epsilon t} \,. \tag{7}$$

Eq. (7) shows that the $\theta(t)$ function has the following Fourier representation:

$$\theta(t) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{i}{\omega + i\epsilon} \tag{8}$$

(c) Differentiate both sides of Eq. (8) with resepect to t and comment on what you find.

Problem 6. Radii of convergence

(a) What is the radius of convergence of the following functions at $z = 2i$.

(i) $\frac{1}{\sqrt{z^2 - 4}}$

(ii) $\sin(z)$. Hint show that $\sin(z)$ is an entire function as discussed below.

(iii) $\log(z + 4)$

(b) Recall that the radius of convergence of a power series is given by the formula[1]

$$\frac{1}{R} = \lim_{n \to \infty} |a_n|^{1/n} \tag{11}$$

Given the power series expansion of the Bessel function from the previous homework, show that the radius of convergence of $J_0(x)$ is infinity using this formula

A function which has an infinite radius of convergence is known as an *entire* function. Examples of entire functions are $\sin(z), \cos(z), e^z, J_n(z)$.

[1] In general this assumes that the limit exists. A pure mathematician might write

$$\frac{1}{R} = \lim_{n \to \infty} \sup |a_n|^{1/n} \tag{9}$$

which means that one should take the maximum of the limit points of the sequence. For instance for the series

$$f(z) = 2^0 + 3^1 z + 2^2 z^2 + 3^3 z^3 + 2^4 z^4 + 3^5 z^5 + \ldots \,, \tag{10}$$

the sequence, $|a_n|^{1/n}$, has two limit points: 2 and 3. The maximum of these limit points is 3 and the radius of convergence is 1/3.

11.2.10 For what complex values do each of the following functions $f(z)$ have a derivative?

(a) $f(z) = z^{3/2}$,
(b) $f(z) = z^{-3/2}$,
(c) $f(z) = \tan^{-1}(z)$,
(d) $f(z) = \tanh^{-1}(z)$.

11.2.11 Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, v) + iv(x, y)$. We label the real part, $u(x, y)$, the velocity potential, and the imaginary part, $v(x, y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic:

(a) Show that $df/dz = V_x - iV_y$.
(b) Show that $\nabla \cdot \mathbf{V} = 0$ (no sources or sinks).
(c) Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).

11.2.12 The function $f(z)$ is analytic. Show that the derivative of $f(z)$ with respect to z^* does not exist unless $f(z)$ is a constant.

Hint. Use the chain rule and take $x = (z + z^*)/2$, $y = (z - z^*)/2i$.

Note. This result emphasizes that our analytic function $f(z)$ is not just a complex function of two real variables x and y. It is a function of the complex variable $x + iy$.

11.3 Cauchy's Integral Theorem

Contour Integrals

With differentiation under control, we turn to integration. The integral of a complex variable over a path in the complex plane (known as a **contour**) may be defined in close analogy to the (Riemann) integral of a real function integrated along the real x-axis.

We divide the contour, from z_0 to z_0', designated C, into n intervals by picking $n - 1$ intermediate points $z_1, z_2, \ldots$ on the contour (Fig. 11.2). Consider the sum

$$S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}),$$

where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with

$$|z_j - z_{j-1}| \to 0$$

for all j. If $\lim_{n\to\infty} S_n$ exists, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)\,dz = \int_C f(z)\,dz. \tag{11.16}$$

The right-hand side of Eq. (11.16) is called the contour integral of $f(z)$ (along the specified contour C from $z = z_0$ to $z = z_0'$).

Exercises

11.3.1 Show that $\int_{z_1}^{z_2} f(z)\,dz = -\int_{z_2}^{z_1} f(z)\,dz.$

11.3.2 Prove that $\left|\int_C f(z)dz\right| \le |f|_{\max}\cdot L,$

where $|f|_{\max}$ is the maximum value of $|f(z)|$ along the contour C and L is the length of the contour.

11.3.3 Show that the integral

$$\int_{3+4i}^{4-3i} (4z^2 - 3iz)\,dz$$

has the same value on the two paths: (a) the straight line connecting the integration limits, and (b) an arc on the circle $|z| = 5$.

11.3.4 Let $F(z) = \int_{\pi(1+i)}^{z} \cos 2\zeta\, d\zeta.$

Show that $F(z)$ is independent of the path connecting the limits of integration, and evaluate $F(\pi i)$.

11.3.5 Evaluate $\oint_C (x^2 - iy^2)\,dz$, where the integration is (a) clockwise around the unit circle, (b) on a square with vertices at $\pm 1 \pm i$. Explain why the results of parts (a) and (b) are or are not identical.

11.3.6 Verify that

$$\int_0^{1+i} z^* dz$$

depends on the path by evaluating the integral for the two paths shown in Fig. 11.7. Recall that $f(z) = z^*$ is not an analytic function of z and that Cauchy's integral theorem therefore does not apply.

11.3.7 Show that

$$\oint_C \frac{dz}{z^2 + z} = 0,$$

in which the contour C is a circle defined by $|z| = R > 1$.

Hint. Direct use of the Cauchy integral theorem is illegal. The integral may be evaluated by expanding into partial fractions and then treating the two terms individually. This yields 0 for $R > 1$ and $2\pi i$ for $R < 1$.

11.7.11 A function $f(z)$ is analytic along the real axis except for a third-order pole at $z = x_0$. The Laurent expansion about $z = x_0$ has the form

$$f(z) = \frac{a_{-3}}{(z - x_0)^3} + \frac{a_{-1}}{z - x_0} + g(z),$$

with $g(z)$ analytic at $z = x_0$. Show that the Cauchy principal value technique is applicable, in the sense that

(a) $\lim_{\delta \to 0} \left\{ \int_{-\infty}^{x_0 - \delta} f(x)\,dx + \int_{x_0+\delta}^{\infty} f(x)\,dx \right\}$ is finite.

(b) $\int_{C_{x_0}} f(z)\,dz = \pm i\pi a_{-1}$,

where C_{x_0} denotes a **small semicircle** about $z = x_0$.

11.7.12 The unit step function is defined as (compare Exercise 1.15.13)

$$u(s - a) = \begin{cases} 0, & s < a \\ 1, & s > a. \end{cases}$$

Show that $u(s)$ has the integral representations

(a) $u(s) = \lim_{\varepsilon \to 0^+} \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \frac{e^{ixs}}{x - i\varepsilon}\,dx.$

(b) $u(s) = \dfrac{1}{2} + \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \frac{e^{ixs}}{x}\,dx.$

Note. The parameter s is real.

11.8 Evaluation of Definite Integrals

Definite integrals appear repeatedly in problems of mathematical physics as well as in pure mathematics. In Chapter 1 we reviewed several methods for integral evaluation, there noting that contour integration methods were powerful and deserved detailed study. We have now reached a point where we can explore these methods, which are applicable to a wide variety of definite integrals with physically relevant integration limits. We start with applications to integrals containing trigonometric functions, which we can often convert to forms in which the variable of integration (originally an angle) is converted into a complex variable z, with the integration integral becoming a contour integral over the unit circle.

Trigonometric Integrals, Range (0,2π)

We consider here integrals of the form

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta, \tag{11.91}$$

where f is finite for all values of θ. We also require f to be a rational function of $\sin\theta$ and $\cos\theta$ so that it will be single-valued. We make a change of variable to

$$z = e^{i\theta}, \quad dz = ie^{i\theta}d\theta,$$

with the range in θ, namely $(0, 2\pi)$, corresponding to $e^{i\theta}$ moving counterclockwise around the unit circle to form a closed contour. Then we make the substitutions

$$d\theta = -i\,\frac{dz}{z}, \quad \sin\theta = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \tag{11.92}$$

where we have used Eq. (1.133) to represent $\sin\theta$ and $\cos\theta$. Our integral then becomes

$$I = -i\oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{z}, \tag{11.93}$$

with the path of integration the unit circle. By the residue theorem, Eq. (11.64),

$$I = (-i)\,2\pi i \sum \text{residues within the unit circle.} \tag{11.94}$$

Note that we must use the residues of f/z. Here are two preliminary examples.

Example 11.8.1 Integral of cos in Denominator

Our problem is to evaluate the definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \quad |a| < 1.$$

By Eq. (11.93) this becomes

$$I = -i \oint_{\text{unit circle}} \frac{dz}{z[1 + (a/2)(z + z^{-1})]}$$

$$= -i\frac{2}{a}\oint \frac{dz}{z^2 + (2/a)z + 1}.$$

The denominator has roots

$$z_1 = -\frac{1 + \sqrt{1 - a^2}}{a} \quad \text{and} \quad z_2 = -\frac{1 - \sqrt{1 - a^2}}{a}.$$

Noting that $z_1 z_2 = 1$, it is easy to see that z_2 is within the unit circle and z_1 is outside. Writing the integral in the form

$$\oint \frac{dz}{(z - z_1)(z - z_2)},$$

we see that the residue of the integrand at $z = z_2$ is $1/(z_2 - z_1)$, so application of the residue theorem yields

$$I = -i\,\frac{2}{a}\cdot 2\pi i\,\frac{1}{z_2 - z_1}.$$

Inserting the values of z_1 and z_2, we obtain the final result

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \quad |a| < 1.$$

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Example 11.8.2 ANOTHER TRIGONOMETRIC INTEGRAL

Consider

$$I = \int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5 - 4\cos\theta}.$$

Making the substitutions identified in Eqs. (11.92) and (11.93), the integral I assumes the form

$$I = \oint \frac{\frac{1}{2}(z^2 + z^{-2})}{5 - 2(z + z^{-1})} \left(\frac{-i\,dz}{z}\right)$$
$$= \frac{i}{4} \oint \frac{(z^4 + 1)\,dz}{z^2 \left(z - \frac{1}{2}\right)(z-2)},$$

where the integration is around the unit circle. Note that we identified $\cos 2\theta$ as $(z^2 + z^{-2})/2$, which is simpler than reducing it first to its equivalent in terms of $\sin z$ and $\cos z$. We see that the integrand has poles at $z = 0$ (of order 2), and simple poles at $z = 1/2$ and $z = 2$. Only the poles at $z = 0$ and $z = 1/2$ are within the contour.

At $z = 0$ the residue of the integrand is

$$\frac{d}{dz}\left[\frac{z^4 + 1}{\left(z - \frac{1}{2}\right)(z-2)}\right]_{z=0} = \frac{5}{2},$$

while its residue at $z = 1/2$ is

$$\left[\frac{z^4 + 1}{z^2(z-2)}\right]_{z=1/2} = -\frac{17}{6}.$$

Applying the residue theorem, we have

$$I = \frac{i}{4}(2\pi i)\left[\frac{5}{2} - \frac{17}{6}\right] = \frac{\pi}{6}.$$

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We stress that integrals of the type now under consideration are evaluated after transforming them so that they can be identified as exactly equivalent to contour integrals to which we can apply the residue theorem. Further examples are in the exercises.

11.8.2 Show that $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}}, \quad a > 1.$

11.8.3 Show that $\int_0^{2\pi} \frac{d\theta}{1-2t\cos\theta+t^2} = \frac{2\pi}{1-t^2}, \quad \text{for } |t| < 1.$

What happens if $|t| > 1$? What happens if $|t| = 1$?

11.8.4 Evaluate $\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$.

ANS. $\pi/12$.

11.8.5 With the calculus of residues, show that

$$\int_0^{\pi} \cos^{2n}\theta\, d\theta = \pi\frac{(2n)!}{2^{2n}(n!)^2} = \pi\frac{(2n-1)!!}{(2n)!!}, \quad n = 0, 1, 2, \ldots.$$

The double factorial notation is defined in Eq. (1.76).

Hint. $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}) = \frac{1}{2}(z+z^{-1}), \quad |z| = 1.$

11.8.6 Verify that simplification of the expression in Eq. (11.112) yields the result given in Eq. (11.113).

11.8.7 Complete the details of Example 11.8.8 by verifying that there is no contribution to the contour integral from either the small or the large circles of the contour, and that Eq. (11.115) simplifies to the result given as (11.116).

11.8.8 Evaluate $\int_{-\infty}^{\infty} \frac{\cos bx - \cos ax}{x^2}\, dx, \quad a > b > 0.$

ANS. $\pi(a-b)$.

11.8.9 Prove that $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$

Hint. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$.

11.8.10 Show that $\int_0^{\infty} \frac{x\sin x}{x^2+1}\, dx = \frac{\pi}{2e}.$