Integrals

a) $$I = \int_0^\infty \left(\frac{\sin x}{x}\right)^2 \longleftarrow$$ the integral is symmetric

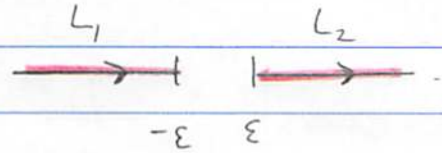
$$= \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{(1-\cos 2x)}{2x^2}$$

Now note that we may replace $\cos 2x$ with e^{i2x} since $e^{i2x} = \cos 2x + i\sin 2x$ and the $\sin 2x$ piece is odd and vanishes upon integration

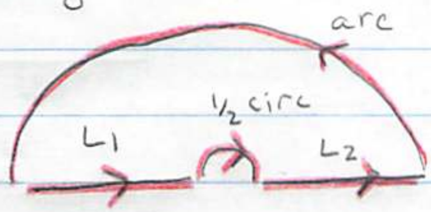
$$\int_{-\infty}^{\infty} \frac{\overset{\text{odd}}{\sin x}}{\underset{\text{even}}{x^2}}\,dx$$

Here this integral is understood as excluding a region of order ε around the origin, i.e. a principal value



$$I_{L_1+L_2} = \frac{i}{4}\int_{L_1+L_2} \frac{1-e^{2ix}}{x^2} \qquad I = \lim_{\varepsilon \to 0} I_{L_1+L_2}$$

Closing the contour,



Noting the arc gives zero,

$$I_C = \oint \frac{1-e^{2iz}}{4z^2} = \int_{L_1+L_2} \frac{1-e^{2iz}}{4z^2}\,dz + \int_{\frac{1}{2}\text{circ}} \frac{1-e^{2iz}}{4z^2}\,dz$$

Now $I_C = 0$ since the integrand is analytic in UHP yielding

$$I_{L_1+L_2} = -\int_{1/2\,\text{circ}} \frac{(1-e^{2iz})\,dz}{4z^2}$$

Expanding near zero

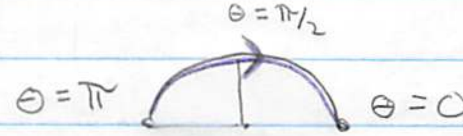
$$-\frac{1}{4z^2}(1-e^{2iz}) = 0 + \frac{i}{2z} + \ldots$$

Then $\frac{dz}{z} = +i\,d\theta$, $z = \varepsilon e^{+i\theta}$, so the integral is

$$I_{L_1+L_2} = \int_{\pi}^{0} +i\,d\theta \left[0 + \frac{i}{2} \right] + O(\varepsilon)$$

$$I_{L_1+L_2} = 0 + \frac{\pi}{2}$$

note the limits

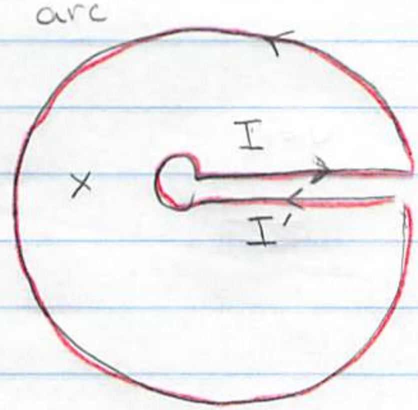


So

$$I = \text{Re } I_{L_1+L_2} = \frac{\pi}{2}$$

b) Then

$$I = \int_0^\infty dx \ \frac{x^{\alpha-1}}{1+x}$$



Now $x^{\alpha-1} \rightarrow e^{i2\pi(\alpha-1)} \dfrac{x^{\alpha-1}}{1+x}$:

So, $I' = -e^{i2\pi(\alpha-1)}$
$$\times \int_0^\infty dx \ \frac{x^{\alpha-1}}{1+x}$$

Then

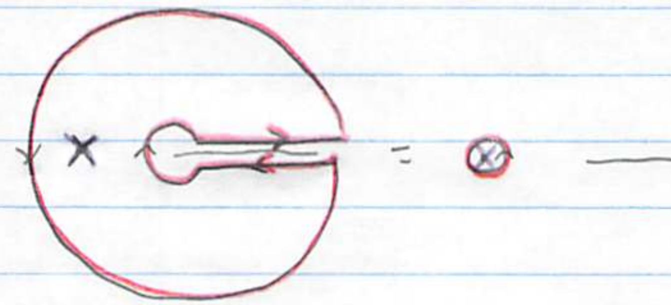
$$I = \frac{1}{(1-e^{i2\pi(\alpha-1)})}(I+I')$$

Now we note for $\alpha > 0$ and $\alpha < 1$ the integral converges. Further the arc at ∞ gives nothing since

$$\int_{\text{arc}} \frac{dz}{z} \ \frac{z^\alpha}{1+z} \xrightarrow[|z|\to\infty]{} \int d\theta \ \frac{R^\alpha}{R} \ e^{i(\alpha-1)} \rightarrow 0$$

Then we deform the arc around the pole

$$I + I' = \oint_{\text{pole}} dz \, \frac{z^{\alpha-1}}{1+z}$$



$$\operatorname*{Res}_{z=e^{i\pi}} \frac{z^{\alpha-1}}{1+z} = e^{i\pi(\alpha-1)}$$

So

$$I = \frac{1}{\left(1 - e^{i2\pi(\alpha-1)}\right)} e^{i\pi(\alpha-1)} \, 2\pi i$$

Or

$$I = \frac{-1}{e^{-\pi i(\alpha-1)} - e^{i\pi(\alpha-1)}} \pi \, 2i$$

$$I = \frac{\pi}{-\sin \pi(\alpha-1)} = \frac{\pi}{\sin \pi\alpha}$$

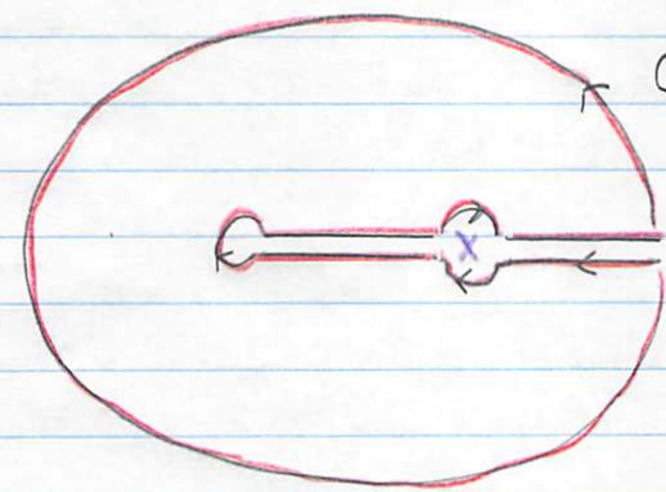
ii) The second integral is closely related

$$I = \int_0^\infty x^{\alpha-1} \frac{P}{1-x^2} = \int_0^\infty x^{\alpha-1} \frac{P}{2(1-x)} + \frac{P}{(1+x)2} = \frac{1}{2}(I_1 + I_2)$$

$$I_1 = \int_0^\infty x^{\alpha-1} \frac{P}{1-x}$$

$$I_2 = \int_0^\infty x^{\alpha-1} \frac{P}{1+x} = \frac{\pi}{\sin \pi\alpha} \leftarrow \text{from the previous problem}$$

for the first integral we use the following contour



Now

$$I_c = \oint \frac{z^{\alpha-1}}{1-z} = 0$$

And

Now

$$I_c = P\int_0^\infty \frac{x^{\alpha-1}}{1-x} + \int_{\curvearrowright} \frac{z^{\alpha-1}}{1-z} + \cdots$$

$$+ - e^{2\pi i(\alpha-1)} \int_0^\infty dx\ x^{\alpha-1}\, P\frac{1}{1-x} + \int_{\curvearrowleft} \frac{z^{\alpha-1}}{1-z}$$

Now

$$I_c = \left(1 - e^{2\pi i(\alpha-1)}\right)\int_0^\infty dx\ x^{\alpha-1}\, \frac{P}{1-x}$$

$$+ \int_{\curvearrowright} \frac{z^{\alpha-1}}{1-z} + \int_{\curvearrowleft} \frac{z^{\alpha-1}}{1-z}\, dz$$

In the first arc $z = e^{i0} + \underbrace{\varepsilon e^{+i\theta}}_{z-1}$ while in the second arc:

$$z = e^{2\pi i} + \underbrace{\varepsilon e^{+i\theta}}_{z-1}$$

Then $d(z-1)/(z-1) = i\,d\theta$ and we integrate $\theta = \pi \ldots 0$

$$\int_{\curvearrowright} \frac{z^{\alpha-1}}{1-z} = -\int_\pi^0 i\,d\theta = i\pi$$

while similarly

$$\int_{\curvearrowleft} \frac{z^{\alpha-1}}{1-z} = -e^{2\pi i(\alpha-1)} \int_0^{-\pi} i\,d\theta = e^{2\pi i(\alpha-1)}\, i\pi$$

Thus

$$I_c = \left(1 - e^{2\pi i(\alpha-1)}\right) \overbrace{\int_0^{\infty} dx\, x^{\alpha-1} \frac{P}{1-x}}^{I_2}$$

$$+ \left(1 + e^{2\pi i(\alpha-1)}\right) i\pi = 0$$

Thus

$$I_2 = -i\pi \frac{\left(1 + e^{2\pi i(\alpha-1)}\right)}{\left(1 - e^{2\pi i(\alpha-1)}\right)} = -\frac{\pi}{-1} \cdot \frac{\cos\pi(\alpha-1)}{\sin\pi(\alpha-1)}$$

$$I_2 = \frac{\cos\pi\alpha}{\sin\pi\alpha}\,\pi$$

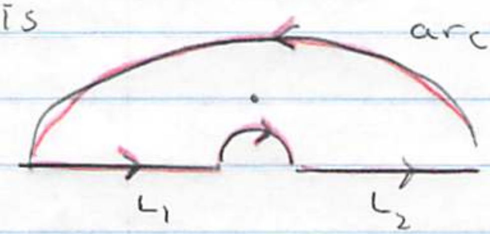
And Finally we find

$$I = \frac{1}{2}(I_1 + I_2) = \frac{\pi}{2} \frac{(1+\cos\pi\alpha)}{\sin\pi\alpha}$$

c) We wish to evaluate

$$I = \int_0^\infty \frac{(\log x)^2}{x^2+1} = \int_{-\infty}^0 dx \frac{(\log |x|)^2}{x^2+1}$$

On the positive real axis $\log z = \log r$ while on the negative real axis



$$\lg z = \log r + i\pi$$

$$(\lg z)^2 = (\log r)^2 + 2i\pi \log r - \pi^2$$

Thus

$$\int_{L_1+L_2} \frac{(\log z)^2}{z^2+1} = I + I + \int_{-\infty}^0 dr \frac{2i\pi \log r - \pi^2}{r^2+1}$$

So

$$\mathrm{Re} \int_{L_1+L_2} \frac{(\log z)^2}{z^2+1} = 2I - \int_{-\infty}^0 \frac{\pi^2}{r^2+1}$$

$$= 2I - \frac{\pi^3}{2}$$

Now it is easy to see that the $\frac{1}{2}\frown$ and the great arc do not contribute

$$I_C = \oint_C dz \frac{(\log z)^2}{z^2+1} = \overset{2\pi i}{\operatorname{Res}_{z=i}} \frac{(\log z)^2}{1+z^2} = \frac{\left(\log \overset{i}{e^{i\pi/2}}\right)^2}{2i} 2\pi i$$

Then

$$I_C = -\frac{\pi^3}{4}$$

Thus

$$-\frac{\pi^3}{4} = 2I - \frac{\pi^3}{2} \implies \boxed{I = \frac{\pi^3}{8}}$$

Atanh

a) Just substitute iy into

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = i\left(\frac{e^{y} - e^{-y}}{2}\right)$$

b) $$\tanh y = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} = y - \frac{y^3}{3}$$

let $y = \text{atanh}\, x$ $\tanh(\text{atanh}\, x) = x$

and let $u = e^{y}$. Then

$$\frac{u - 1/u}{u + 1/u} = x \quad \text{solve for } u^2$$

$$u^2 = \frac{1+x}{1-x} = e^{2y} \Rightarrow y = \frac{1}{2}\log\frac{(1+x)}{(1-x)}$$

c) $$y = \frac{1}{2}\left[x - \frac{x^2}{2} + \frac{x^3}{3} +\right] - \frac{1}{2}\left[-x - \frac{x^2}{2} - \frac{x^3}{3}\right]$$

$$y = x + \frac{x^3}{3} + \ldots$$

d) $$\tanh(\text{atanh}\, x) \simeq \left(x + \frac{x^3}{3} + \ldots\right) - \left(x + x^3/3 + \ldots\right)^3/3$$

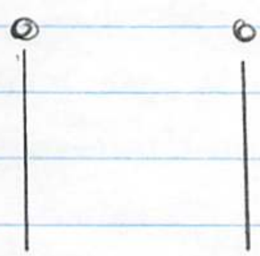
$$\simeq x + \frac{x^3}{3} - \frac{x^3}{3} = x + O(x^5)$$

e) Then, let $x' = \tanh u$, $dx' = \text{sech}^2 u$, $\frac{1}{1-u^2} = \frac{1}{\text{sech}^2}$

$$\int_0^x \frac{dx'}{1-(x')^2} = \int_0^{\text{atan}\,x} du = \text{atan}\,x$$

f) A

B

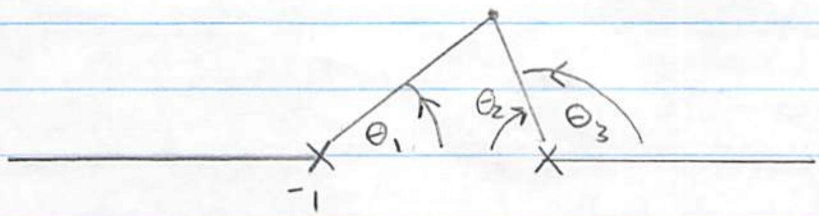


C.

For A

$$\operatorname{atanh} z = \frac{1}{2}\log(1+z) - \frac{1}{2}\log(1-z) \qquad (\text{Eq} \star)$$

We define angles as follows



$$1+z = \rho_1 e^{i\theta_1} \qquad -\pi < \theta_1 \le \pi$$

$$z-1 = \rho_2 e^{i\theta_3} \qquad 0 < \theta_3 \le 2\pi \qquad \theta_2 = \pi - \theta_3 \quad [-\pi .. \pi]$$

$$\log(1+z) = \log\rho_1 + i\theta_1$$

$$\log(1-z) = \log(z-1) + \log(-1)$$

$$= \log\rho_2 + i(\theta_3) - i\pi$$

$$= \log\rho_2 + i\underbrace{(\theta_3 - \pi)}_{-\theta_2}$$

we chose $\log(-1) = -i\pi$ so that atanh z agrees with $z + z^3/3$ at small z

Then from Eq $\star$

$$\operatorname{atanh} z = \frac{1}{2}\log\frac{\rho_1}{\rho_2} + \frac{i(\theta_1 + \theta_2)}{2}$$

For A

	atanh A
$0 + i\varepsilon$	0
$0 - i\varepsilon$	0
$3 + i\varepsilon$	$\frac{1}{2}\log(4/2) + \frac{i\pi}{2}$
$3 - i\varepsilon$	$\frac{1}{2}\log(4/2) - \frac{i\pi}{2}$
$1 - 2i + \varepsilon$	$\frac{1}{2}\log\sqrt{8}/2 + \frac{i}{2}(-\pi/4 - \pi/2)$
$1 - 2i - \varepsilon$	$1/2 \log\sqrt{8}/2 + \frac{i}{2}(-\pi/4 - \pi/2)$
$-3 + i\varepsilon$	$\frac{1}{2}\log(2/4) + \frac{i}{2}(\pi + 0)$
$-3 - i\varepsilon$	$\frac{1}{2}\log(2/4) + \frac{i}{2}(-\pi + 0)$

For B

$$\text{atanh} = \frac{+1}{2} \log \frac{\rho_1}{\rho_2} + i\left(\frac{\Theta_1 + \Theta_2}{2}\right)$$

but Θ_1 $[-\pi/2 \ldots 3\pi/2]$ and Θ_2 is in $[-\pi/2 \ldots 3\pi/2]$

$0 + i\varepsilon$	0
$0 - i\varepsilon$	0
$3 + i\varepsilon$	$\frac{1}{2} \log(4/2) + \frac{i\pi}{2}$
$3 - i\varepsilon$	$\frac{1}{2} \log(4/2) + i\pi/2$
$1 - 2i + \varepsilon$	$1/2 \log(\sqrt{8}/2) + \frac{i}{2}(-\pi/4 + 3\pi/2)$
$1 - 2i - \varepsilon$	$1/2 \log(\sqrt{8}/2) + i/2(-\pi/4 - \pi/2)$
$-3 + i\varepsilon$	$\frac{1}{2} \log(2/4) + \frac{i}{2}(\pi + 0)$
$-3 - i\varepsilon$	$\frac{1}{2} \log(2/4) + \frac{i}{2}(\pi)$

For C

$-\pi \le \theta_1 < \pi$ $\qquad$ $0 < \theta_2 < 2\pi$

$0 + i\varepsilon$	0
$0 - i\varepsilon$	$i\pi$
$3 + i\varepsilon$	$\frac{1}{2}\log(4/2) + \frac{i\pi}{2}$
$3 - i\varepsilon$	$\frac{1}{2}\log(4/2) + i\pi$
$1 - 2i + \varepsilon$	$\frac{1}{2}\log(\sqrt{8}/2) + \frac{i}{2}(-\pi/4 + 3\pi/2)$
$1 - 2i - \varepsilon$	$\frac{1}{2}\log(\sqrt{8}/2) + \frac{i}{2}(-\pi/4 + 3\pi/2)$
$-3 + i\varepsilon$	$\frac{1}{2}\log(2/4) + \frac{i}{2}(\pi)$
$-3 - i\varepsilon$	$\frac{1}{2}\log(2/4) + \frac{i}{2}(-\pi + 2\pi)$