g) Then $y = \operatorname{asin} x$

$$\sin(y) = \frac{e^{iy} - e^{-iy}}{2i} = x$$

let $u = e^{iy}$

$$\frac{u - 1/u}{2i} = x, \quad \text{solve for } u = ix \pm \sqrt{1-x^2}$$

Then

$$y = \frac{1}{i} \log\left(ix \pm \sqrt{1-x^2}\right)$$

we choose the + root so that at small x

$$y \simeq x \qquad \text{i.e. } x = \sin(y) \simeq \sin(x)$$

h) Just substitute $x' = \sin u$

$$I(x) = \int_0^x \frac{dx'}{(1-(x')^2)^{1/2}} = \int_0^{\operatorname{asin}(x)} du = \operatorname{asin} x$$

for $x = 1$ $\operatorname{asin}(1) = \pi/2$

i) $$I(x) = \int_1^x \frac{dx'}{\sqrt{x'^2 - 1}}$$

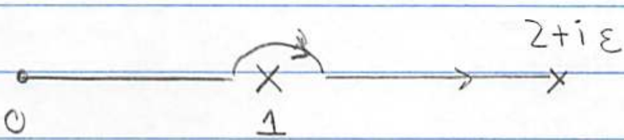
Substitute $\quad x' = \cosh u$

$$I(u) = \int_{\operatorname{acosh}(1)}^{\operatorname{acosh} u} du = \operatorname{acosh}(u) = \log\left(x + \sqrt{x^2 - 1}\right)$$

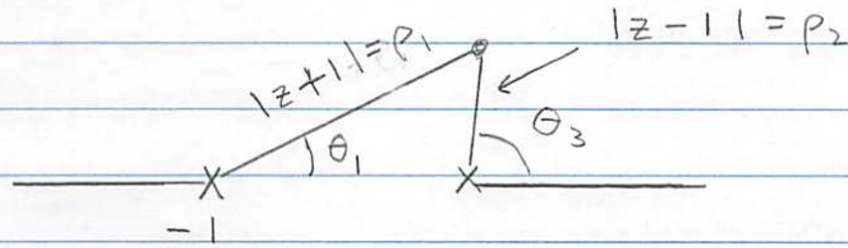
j) Then

$$\operatorname{asin}(z) = \int_0^{2+i\varepsilon} dz' \frac{1}{(1-(z')^2)^{1/2}}$$

The integration is along the path



We need to determine the integrand on the top side of the cut



$\theta_3 = [0, 2\pi)$

$\theta_1 = [-\pi, \pi]$

$$(1-z^2)^{1/2} = (1-z)^{1/2}(z-1)^{1/2}(-1)^{1/2}$$

$$= \rho_1 \rho_2 e^{i\theta_1/2} e^{i\theta_3/2} e^{-i\pi/2}$$

The phase $e^{-i\pi/2}$ is chosen so that near $z=0$ $(\theta_1 = 0, \theta_3 = \pi)$ we get a positive real number consistent with the series of part (g)

Then for $z = x + i\varepsilon$, $\theta_1 = 0$, $\theta_3 = 0$, $x > 1$

$$\sqrt{1-z^2} = (1+x)^{1/2} (x-1)^{1/2} e^{i0/2} e^{i0/2} e^{-i\pi/2}$$

$$= \sqrt{x^2-1}\,(-i)$$

So $\operatorname{asin}(2+i\varepsilon)$

$$= \int_0^1 \frac{dx}{(1-x^2)^{1/2}} + \int_1^2 \frac{dx}{(x^2-1)^{1/2}} \frac{1}{(-i)}$$

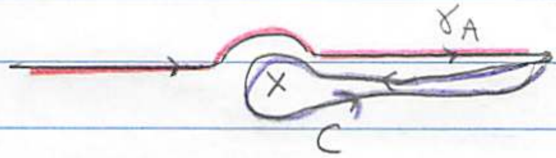
$$= \pi/2 + i \log\left(x + \sqrt{x^2-1}\right)\Big|_1^2$$

$$= \pi/2 + i \log(2 + \sqrt{3})$$

Now we define asin along various paths we procede by example and then summarize:

(1) Take γ_B

$$I_B = \operatorname{asin}(2+i\varepsilon, \gamma_B) = \operatorname{asin}(2+i\varepsilon, \gamma_A) + \underbrace{\oint_C \frac{1}{(1-\underline{z}^2)} \, \underline{dz}}_{I_c}$$



Then $$I_c = -2 \int_1^2 \frac{1}{(x^2-\underline{1})^{1/2}(-i)}$$

winding # around $z=-1$

$$= -2\log(2+\sqrt{3})\, i \quad \leftarrow \quad W_1 = 1 \quad W_{-1} = 0$$

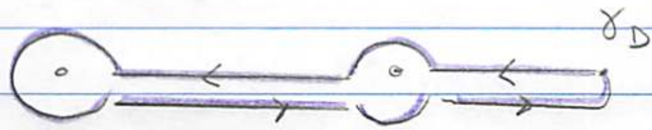
so

Winding number around $z = 1$

$$I_B = \pi/2 - i\log(2+\sqrt{3})$$

(2)

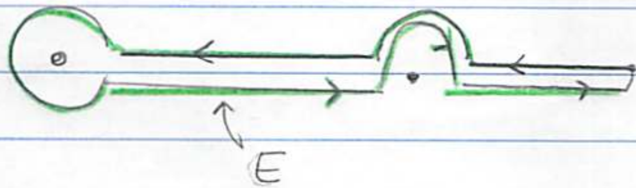
+



$$\oint_{\gamma_D} \frac{1}{(1-\underline{z}^2)} \, \underline{dz} = -2\pi \qquad W_1 = 1 \quad W_{-1} = 1$$

3

As one final example

$\gamma_A +$  has $W_{-1} = 1$ $W_1 = 0$

$$\gamma_E = \oint_E \frac{1}{\sqrt{1-z^2}} = -2i \int_1^2 \frac{1}{(x^2-1)} \mp 2\pi$$

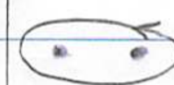
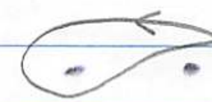
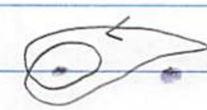
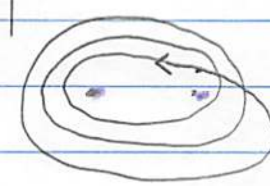
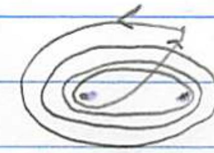
So

$$\text{asin}(2+i\varepsilon, \gamma_A + \gamma_C) = \frac{\pi}{2} - 2\pi - i\log(2+\sqrt{3})$$

Leading to a general formula

$$\text{asin}(2+i\varepsilon) = \frac{\pi}{2} \pm i\log(2+\sqrt{3}) \mp m\,2\pi$$

we give examples below.

	Value ($X \equiv \log(2+\sqrt{3})$)	Winding 1	Wind −1	Path
	$\frac{\pi}{2} + iX$	0	0	
1	$\frac{\pi}{2} + iX \quad - 2iX$	1	0	
	$\frac{\pi}{2} + iX \quad + 0iX$	2	0	
	$\frac{\pi}{2} + iX \quad - 2iX$	−1	0	
2	$\frac{\pi}{2} + iX - 2\pi$	1	1	
	$\frac{\pi}{2} + iX + 2\pi$	−1	−1	
3	$\frac{\pi}{2} + iX \; - 2\pi \; - 2iX$	0	1	
	$\frac{\pi}{2} + iX - 2\pi - 2iX$	0	−1	
	$\frac{\pi}{2} + iX$	0	2	
	$\frac{\pi}{2} + iX$	0	−2	
	$\pi/2 + iX \quad - 3(2\pi)$	3	3	
	$\pi/2 + iX - 3\pi - 2\pi - 2iX$	3	4	
	$\pi/2 + iX \quad - 3\pi$	3	5	

l) We have

$$\operatorname{acosh} x = \log\left(x + \sqrt{x^2-1}\right)$$

$$\frac{d(\operatorname{acosh} x)}{dx} = \frac{1}{(x^2-1)^{1/2}} = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)$$

So

$$\operatorname{acosh}(z,\gamma) = \int_0^z \frac{1}{(z^2-1)^{1/2}}$$

Kramer's Kronig

a)

$$\mathrm{Re}\, G_R = P \int \frac{d\omega'}{\pi} \frac{\mathrm{Im}\, G_R(\omega')}{\omega' - \omega}$$

$$\mathrm{Im}\, G_R = -P \int \frac{d\omega'}{\pi} \frac{\mathrm{Re}\, G_R(\omega')}{\omega' - \omega}$$

Then

$$G_R(\omega) = \int_0^\infty e^{i\omega\tau} G_R(\tau)$$

Taking the condgate

$$G_R^*(\omega) = \int_0^\infty e^{-i\omega\tau} G_R(\tau) = G_R(-\omega)$$

This says

$$\mathrm{Re}\, G_R(\omega) = \mathrm{Re}\, G_R(-\omega) \quad \text{even}$$

$$-\mathrm{Im}\, G_R(\omega) = \mathrm{Im}\, G_R(-\omega) \quad \text{odd}$$

Using these properties from 1

$$\mathrm{Re}\, G_R(\omega) = P \int_{-\infty}^{0} \frac{d\underline{\omega}}{\pi} \frac{\overset{\text{odd}}{\mathrm{Im}\, G_R(\underline{\omega})}}{\underline{\omega} - \omega} + P \int_0^\infty \frac{d\omega'}{\pi} \frac{\mathrm{Im}\, G_R(\omega')}{\omega' - \omega}$$

now change vars $\underline{\omega} = -\omega'$. The first integral becomes

$$P \int_0^\infty \frac{d\omega'}{\pi} \frac{\mathrm{Im}\, G_R(-\omega')}{-\omega' - \omega} = P \int_0^\infty \frac{d\omega'}{\pi} \frac{\mathrm{Im}\, G_R(\omega')}{\omega' + \omega}$$

which implies

$$\mathrm{Re}\, G_R(\omega) = P \int_0^\infty \frac{d\omega'}{\pi} \; \mathrm{Im}\, G_R(\omega') \left[\frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} \right]$$

$$= 2P \int_0^\infty \frac{d\omega'}{\pi} \; \frac{\mathrm{Im}\, G_R(\omega')}{(\omega')^2 - \omega^2}$$

Similarly

$$\mathrm{Im}\, G_R(\omega) = -P \int_{-\infty}^{0} \frac{d\omega'}{\pi} \; \frac{\mathrm{Re}\, G_R(\omega')}{\omega' - \omega} + -P \int_0^\infty \frac{d\omega'}{\pi} \; \frac{\mathrm{Re}\, G(\omega')}{\omega' - \omega}$$

$$= -P \int_0^\infty \frac{d\omega'}{\pi} \; \mathrm{Re}\, G_R(\omega) \left[\frac{1}{\omega' - \omega} - \frac{1}{\omega' + \omega} \right]$$

$$= -\frac{2\omega}{\pi} \int_0^\infty d\omega' \; \frac{\mathrm{Re}\, G_R(\omega')}{(\omega')^2 - \omega^2}$$

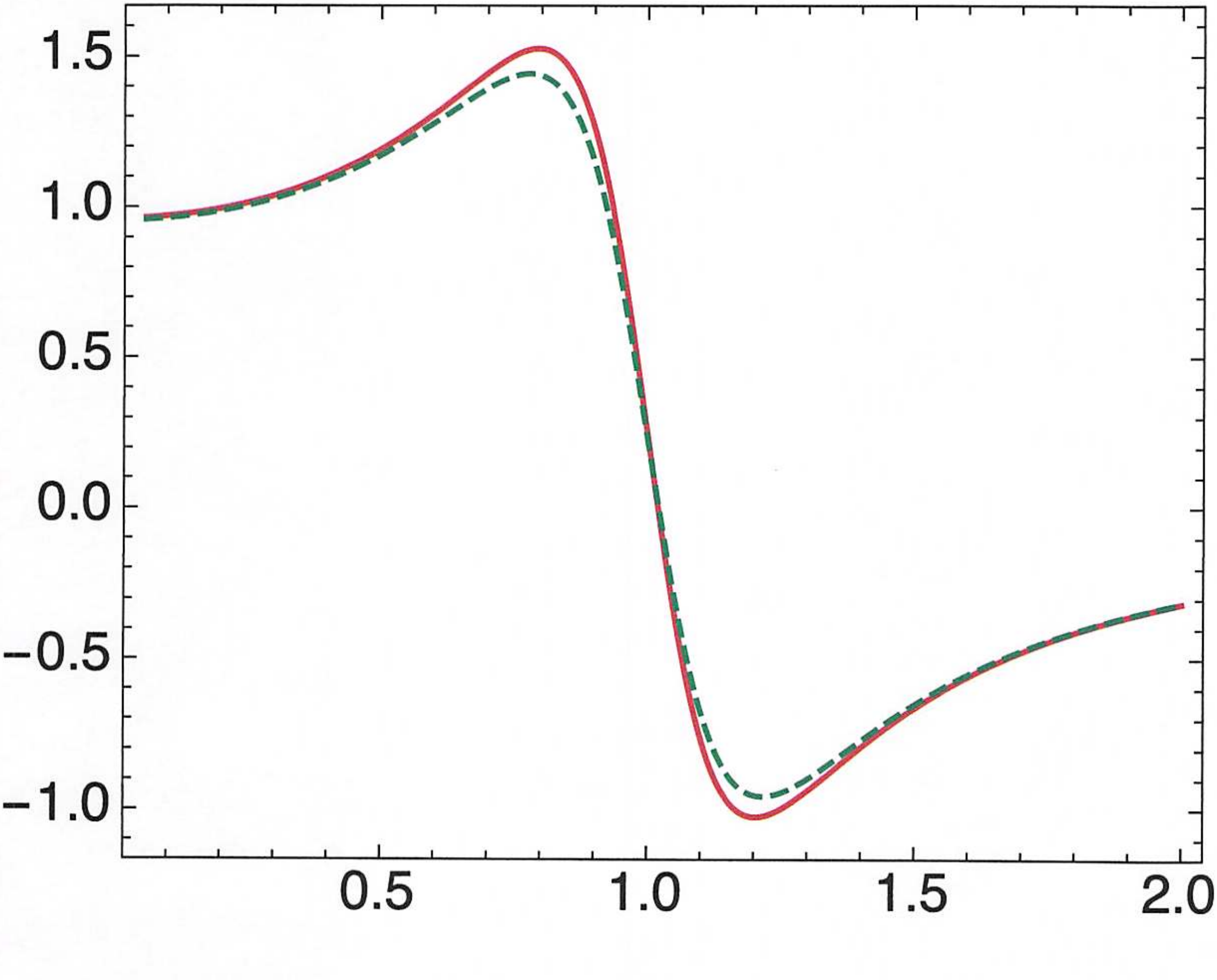
```
(* These are the parameters *)
w0 = 1.
g = 0.2
(* You can make epsilon much smaller. I leave it
this large so that one can see the effect of a finite epsilon
on the plot*)
epsilon = 0.002

(* This is the real part of the function *)
ref[x_?NumericQ] :=  -1/2 (x- w0)/((x-w0)^2 + g^2)  +  1/2 (x + w0)/( (x+w0)^2 + g^2
)

(* This is the imaginary part of the function *)
imf[x_?NumericQ] :=   1/2 g /((x-w0)^2 + g^2)  - 1/2 g /((x+w0)^2 + g^2)

(* This is the real part of the function computed by integrating the imaginary part,
 while carefully avoiding the singularity. We integrate from
zero up to y-epsilon, and y+epsilon to Infinity *)
refn[x_?NumericQ]  := 2/Pi NIntegrate[ y  imf[y]  1/(y^2 - x^2), {y, 0, x-10*epsilon
}]  + 2/Pi NIntegrate[ y imf[y] 1/(y^2 - x^2), {y, x+10*epsilon, Infinity} ]
(* Now I make a plot of the two  functions for comparison*)
gr1 = Plot[  {ref[x],  refn[x]}  , {x,0.06, 2}, PlotStyle->{ls1, ls2}  ]

(* Print out the graph for the homework *)
Export["gr1.eps", gr1]
```


1.5
1.0
0.5
0.0
−0.5
−1.0
0.5
1.0
1.5
2.0