Problem 1. Green functions

(a) Determine the green function of the following differential equation between $x \in [0, a]$ and use it to solve

$$\left[-\frac{d^2}{dx^2} + k^2\right] y = 1 \qquad y(0) = y(a) = 0 \tag{1}$$

A similar equation arises hen determining the potential from a point charge between two grounded parallel plates.

(b) Determine the green function of the following differential equation between $r \in [0, R]$

$$\left[\frac{-1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2}\right] G(r, r_0) = \delta(x-x_0) \qquad y(r) \text{ regular at origin and } y(R) = 0. \tag{2}$$

which arises when finding the potential from a ring of charge inside a grounded sphere. Here $\ell = 0, 1, 2, 3, \ldots$ is a *non-negative integer*.

(c) **Not graded no redo**

(i) Determine a green function which vanishes for $x_0 > x$ for the following differential equation:

$$\left[x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4\right] G(x, x_0) = \delta(x - x_0) \qquad G(x, x_0) = 0 \text{ for } x_0 > x \tag{3}$$

Use it to find a general solution to the following differential equation

$$\left[x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4\right] y = \mathcal{K}x^a \tag{4}$$

where a is $a > 2$. You should find

$$y_p(x) = \frac{\mathcal{K}x^a}{a^2 - 4} \tag{5}$$

By analytically continuing in a we may use $y_p(x)$ for $a < 2$. (You may use mathematica to do definite integrals)

(ii) Show that for $a \to 2 + \epsilon$, that the general solution takes the form

$$y(x) = C_1(\epsilon)x^2 + \frac{C_2}{x^2} + \frac{\mathcal{K}}{4}x^2\log(x) + \mathcal{O}(\epsilon) \tag{6}$$

Thus, the particular solution can be taken as

$$y_p(x) = \frac{\mathcal{K}}{4}x^2\log(x) \tag{7}$$

in this case.

(iii) Finally, consider

$$\left[x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4\right] y = \mathcal{K} x^a \log(x) \tag{8}$$

where $Re(a) > 2$. Show that the particular solution takes the form

$$y_p(x) = \left(\frac{\mathcal{K}}{a^2-4}\right) x^a \log(x) + x^a \frac{\partial}{\partial a}\left(\frac{\mathcal{K}}{a^2-4}\right) \tag{9}$$

$$= \left(\frac{\mathcal{K}}{a^2-4}\right) x^a \log(x) + x^a \left(\frac{-2a}{(a^2-4)^2}\right) \tag{10}$$

(You may use mathematica to do definite integrals.) These are useful when determining the perturbative expansion of the modified Bessel functions $I_2(x)$ and $K_2(x)$ near zero – see the problem below.

Problem 2. Damped harmonic oscillator

Consider the equation of motion of a damped harmonic oscillator for times $t_0[-a, a]$

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2\right] y(t) = f(t) \text{with } x(t) = 0 \text{ for } t < -a. \tag{11}$$

(a) Determine the two homogeneous solutions to the differential equation.

$$y_1 = C_1 e^{-i\omega_1 t} + C_2 e^{-i\omega_2 t}$$

Do not assume that η is small.

(b) What are the two homogeneous solutions to the differential equation when $\eta/2 = \omega_o$ when $\omega_1 = \omega_2$. Notice that the non-trivial second homogeneous solution is

$$\left.\frac{\partial y_1}{\partial \omega}\right|_{\omega=\omega_1} \tag{12}$$

Without introducing Fourier transforms determine the Green function to the differential equation in the limit $\eta/2 = \omega_o$.

(c) By taking Fourier transforms, show that the retarded Green function can be written for generic η as

$$G_R(t, t_o) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{-e^{-i\omega(t-t_o)}}{m(\omega-\omega_1)(\omega-\omega_2)} \tag{13}$$

(d) Perform the Fourier transform in Eq. (13) using complex analysis and determine the retarded Green function.

(e) Consider the limit $\omega_1 = \omega_2$ where the retarded Green function takes the form

$$G_R(t, t_o) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{-e^{-i\omega(t-t_o)}}{m(\omega-\omega_1)^2} \tag{14}$$

Determine the green function by Fourier transform in this case, and show that it reproduces the results of part (b).

Problem 3. Classify

Classify all the singular points (both at $x = 0$ and $x \to \infty$ on the real axis) of the following differential equations. Check your results, by examining the properties of these equations at the DLMF site.

(a) Airy equation

$$y'' = xy \tag{15}$$

(b) Hypergeometric equations

$$x(1-x)y'' + (c - (a+b+1)x)y' - ab\, y = 0 \tag{16}$$

(c) Kummer's confluent hypergeometric equation

$$xy'' + (b-x)y' - ay = 0 \tag{17}$$

For all regular singular points determine the leading behavior close to the singularity

$$y = C_1 x^{s_1} + C_2 x^{s_2} \tag{18}$$

with specific s_1 and s_2. For all irregular singular points, determine the asymptotic forms

$$y = C_1 e^{S_1(x)} + C_2 e^{S_2(x)} \tag{19}$$

Problem 4. Series solution near a regular point

(a) Find the Taylor expansion about $x = 0$ of the solution to the initial-value problem

$$(x-1)(x-2)y'' + (4x-6)y' + 2y = 0 \qquad y(0) = 1 \text{ and } y'(0) = 1 \tag{20}$$

You should find a recursion relation for a series solution $\sum a_n x^n$ of the form

$$a_{n+2} = \frac{-1}{2} a_n + \frac{3}{2} a_{n+1} \tag{21}$$

and can use this to determine $y(x)$ explicitly.

(b) On general grounds (i.e. without $y(x)$ explicitly given) where can you expect the series of part (a) to converge? Explain. Does your explicit series match this expectation?

Problem 5. Series expansion of Modified Bessel Functions

Take the sinusoidal differential equation, which leads to sinusoidal solutions

$$\left[\frac{d^2}{dx^2} + \nu^2\right] y = 0 \qquad y = Ce^{\pm i\nu x} \tag{22}$$

Changing $x \to iz$ leads to a modified differential equation and solutions

$$\left[-\frac{d^2}{dz^2} + \nu^2\right] y = 0 \qquad y = Ce^{\pm \nu z} \tag{23}$$

Similarly Bessel functions satisfy the differential equation

$$\left[x\frac{d}{dx}\left(x\frac{d}{dx}\right) + (x^2 - \nu^2)\right] y = 0 \tag{24}$$

leading to the oscillatory behavior analyzed previously $J_\nu(x)$ and $J_{-\nu}(x)$. Changing $x \to iz$ leads to the modified Bessel equation and the solutions are characterized by exponential growth and decay

$$\left[z\frac{d}{dz}\left(z\frac{d}{dz}\right) - (z^2 + \nu^2)\right] y = 0 \tag{25}$$

where the exponentially growing solution is $I_\nu(z)$ (the modified Bessel function of the first kind), and the exponentially decaying solution $K_\nu(z)$ (the modified Bessel function of the second kind). We will develop a series expansion for $I_2(z)$ and $K_2(z)$

(a) (**do not turn in**) Using `Mathematica` make a graph of the I_0, I_1, I_2 and K_0, K_1, K_2 for $z = 0 \ldots \infty$.

(b) As done in class, near $z = 0$ we approximate $y(z)$ with a series of the following form

$$y(z) = y^{(0)}(z) + y^{(1)}(z) + y^{(2)}(z) + \ldots \tag{26}$$

where each higher order is suppressed by a power of z (up to logs). Determine a hierarchy of equations which determines $y^{(n)}(z)$ from the lower order solutions.

(c) Show that the general form of the zeroth order solution is

$$y^{(0)}(z) = C_1 z^2 + \frac{C_2}{z^2} \tag{27}$$

(d) Start with a $C_2 = 0$, solve the hierarchy of part (b), and show that the solution can be written with $\nu = 2$

$$y_1(x) = C_1 I_\nu(z) = C_1(\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu + k + 1)}. \tag{28}$$

(e) **not graded – not part of redo**. Now set $C_1 = 0$, solve the hierarchy of part (b) with $C_2 = 2$ for later convenience. Here you should use the results of problem $(1.c)$ and show that a second solution takes the form

$$y_2(x) = \left[-\log(x)y_1(x) + \frac{2}{x^2} - \frac{1}{2} + \frac{3x^2}{32} + \frac{17x^4}{1152} + \ldots\right] \tag{29}$$

(f) **not graded – not part of redo**. Show that the asymptotic form of the (modified) Bessel equation takes the form as $x \to \infty$ on the positive real axis

$$y(x) = \beta_1 u_+(x) + \beta_2 u_-(x) \tag{30}$$

where $u_\pm = \frac{e^{\pm x}}{\sqrt{2\pi x}}$

(g) **not graded – not part of redo**. Instead of working with $y_2(x)$ we may work with a linear combination of y_1 and y_2

$$\bar{y}_2(x) = \alpha y_1(x) + y_2(x) \tag{31}$$

as a linearly independent second solution. The constant α can be dialed to ensure certain desirable behavior as $x \to \infty$. As $y_1(x)$ and $y_2(x)$ are evolved (e.g. numerically) from $x = 0$ to $x = \infty$ they can be written for large x

$$y_1 \to C_{1+} u_+ + C_{1-} u_- \tag{32}$$

$$y_2 \to C_{2+} u_+ + C_{2-} u_- \tag{33}$$

Then taking $\alpha = -\frac{C_{2+}}{C_{1+}}$ we can determine a second solution $\bar{y}_2$ which *decreases* exponentially as $x \to \infty$, instead of growing as $x = \infty$. The appropriate value of α turns out to be

$$\alpha = \gamma_E - \log(2) \tag{34}$$

where $\gamma_E = 0.577216$ is the Euler-Mascheroni constant, and stems from the taylor series expansion of the Γ function

$$\Gamma(1 + x) = 1 - \gamma_E x + \frac{1}{12}\left(6\gamma_E^2 + \pi^2\right) x^2 + O\left(x^3\right) \tag{35}$$

then

$$\bar{y}_2(x) \equiv K_2(x) = -(\log(x/2) + \gamma_E) I_2(x) + \frac{2}{x^2} - \frac{1}{2} + \frac{3x^2}{32} + \ldots \tag{36}$$

Using `Mathematica`'s `NDSolve` solve the differential equation numerically. Start at modestly small x (say $x = 0.2$) where the series expansion works well. Use the series expansion to set the initial conditions for a given α. Then evolve the differential equation to $x = 8$. Plot your numerical results for $\alpha = \gamma_E - \log(2)$, and for values that differ somewhat (both positive and negative) from this value. Note that $\gamma_E =$ `EulerGamma` in Mathematica