Problem 1

a) The two homogeneous solutions are $\cosh x$ and $\sinh x$

- We will take $y_<(x) = \sinh x$ and $y_>(x) = \sinh(x-a)$

- You could make other choinces but these satisfy the B.c.

$$G(x,x_0) = C\Big[\sinh x \sinh(x_0-a)\,\Theta(x_0-x) + \sinh(x-a)\sinh x_0\,\Theta(x-x_0)\Big]$$

- This satisfies the b.c. $G(x,x_0) \xrightarrow[x\to 0]{} 0$ and $G(x,x_0) \xrightarrow[x\to a]{} 0$

- From the jump condition

$$\left[-\frac{d^2}{dx^2} + k^2\right] G(x,x_0) = \delta(x-x_0)$$

i.e.

$$-\frac{d\,G(x,x_0)}{dx}\Bigg|_{x=x_0+\varepsilon} + \frac{d\,G(x,x_0)}{dx}\Bigg|_{x=x_0-\varepsilon} = 1$$

We have

$$C\left[-\cosh(x_0-a)\sinh x_0 + \cosh x_0 \sinh(x_0-a)\right] = 1$$

Or using "trig identities"

$$C\left[\sinh\left((x_0-a)-x_0\right)\right]=1$$

Leading to

$$G(x,x_0) = \frac{-\left[\sinh x \sinh(x_0-a)\,\theta(x_0-x)+\sinh(x-a)\sinh x_0\,\theta(x-x_0)\right]}{\sinh a}$$

Then

$$y_p(x) = \int_0^a dx_0 \; G(x,x_0)$$

$$= -\int_0^x dx_0 \; \frac{\sinh(x-a)\sinh x_0}{\sinh a} + -\int_x^a dx_0 \; \frac{\sinh x \; \mathrm{sh}(x_0-a)}{\mathrm{sh}\,a}$$

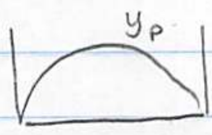
$$y_p(x) = -\frac{\sinh(x-a)(\cosh(x)-1)}{\mathrm{sh}\,a} - \frac{\mathrm{sh}\,x\,(1-\cosh(x-a))}{\mathrm{sh}\,a}$$

$$y_p(x) = \frac{\sinh(a-x)\,(\cosh x - 1)}{\mathrm{sh}\,a} + \frac{(\cosh(x-a)-1)\,\mathrm{sh}\,x}{\mathrm{sh}\,a}$$

A few keystrokes verifies that

$$-y_p'' + y_p = 1$$

It also satisfies the b.c. $y(0)=y(a)=0$



b) • We solve the Green function:

$$\left[-\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2} \right] G(r, r_0) = \delta(r - r_0)$$

• We look for solutions to the homogeneous eqn

$$y = r^s$$

And find that $-s(s+1) + \ell(\ell+1) = 0$, i.e.

$$y_{homo}(r) = C_1 r^{\ell} + \frac{C_2}{r^{\ell+1}}$$

• We require our green fcn to be reg at $r=0$, vanish at $r=r_0$ and continuous at $r=r_0$

$$y_<(r) = \left(\frac{r}{R}\right)^{\ell}$$

$$y_>(r) = -\left(\frac{r}{R}\right)^{\ell} + \left(\frac{R}{r}\right)^{\ell+1}$$

$$G(r, r_0) = C\, y_<(r_<)\, y_>(r_>)$$

• Integrating, we find the jump condition:

$$-r^2 \partial/\partial_r G \Big|_{r=r_0+\varepsilon} + r^2 \partial/\partial_r G(r, r_0) \Big|_{r=r_0-\varepsilon} = r_0^2$$

Or

$$C\left[-r_0^2\, y_>'(r_0)\, y_<(r_0) + r_0^2\, y_<'(r_0)\, y_>(r_0)\right] = r_0^2$$

$$C\,(2l+1)\,R = r_0^2$$

$$C = \frac{r_0^2}{(2l+1)R}$$

So

$$G(r,r_0) = C\left[y_<(r)\, y_>(r_0)\, \Theta(r_0-r) + y_>(r)\, y_<(r_0)\, \Theta(r-r_0)\right]$$

$$= \frac{r_0^2}{(2l+1)}\left[\frac{r^l}{r_0^{l+1}}\left(1-\left(\frac{r_0}{R}\right)^{2l+1}\right)\Theta(r_0-r)\right.$$

$$\left. + \frac{r_0^l}{r^{l+1}}\left(1-\left(\frac{r}{R}\right)^{2l+1}\right)\Theta(r-r_0)\right]$$

$$G(r,r_0) = \frac{r_0^2}{(2l+1)}\left[\frac{r_<^l}{r_>^{l+1}}\left(1-\left(\frac{r_>}{R}\right)^{2l+1}\right)\right]$$

Damped Oscillator

a) Substitute $Ce^{-i\omega t}$

$$\left[-m\omega^2 + m\gamma(-i\omega) + m\omega_0^2\right] Ce^{i\omega t} = 0$$

So we have non-trivial solutions if

$$\left[-m\omega^2 + m\gamma(-i\omega) + m\omega_0^2\right] = 0$$

Thus

$$\omega = -i\frac{\eta}{2} \pm \sqrt{-\left(\frac{\eta}{2}\right)^2 + \omega_0^2} \equiv \omega_\pm$$

b) Then the retarded green fcn is

$$G(t, t_0) = \left[y(t)\frac{\partial y}{\partial \omega}(t_0) - \frac{\partial y}{\partial \omega}(t)\, y(t_0)\right] C\,\theta(t-t_0)$$

which is the super-position of $y(t)$ and $\frac{\partial y}{\partial \omega}(t)$ which vanishes as $t \to t_0$.
It needs to vanish for $t \to t_0$ to be continuous. From

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2\right] G(t, t_0) = \delta(t-t_0)$$

We have

$$m\frac{dG}{dt}\bigg|_{t=t_0+\varepsilon} - \underbrace{m\cancel{\frac{dG}{dt}}}_{0}\bigg|_{t=t_0-\varepsilon} = 1$$

- Instead of

$$y_1 = e^{-i\omega t} \quad \text{and} \quad \frac{\partial y}{\partial \omega} = -it\, e^{-i\omega t}$$

with $\omega = -i\eta/2$ we may use

$$y_1 = e^{-\eta/2 (t - t_0)}$$

$$y_2 = (t - t_0)\, e^{-\eta/2 (t - t_0)}$$

- Then since $G(t, t_0)$ vanishes as $t \longrightarrow t_0$ we must have

$$G(t, t_0) = C\,(t - t_0)\, e^{-\eta/2 (t - t_0)}\, \theta(t - t_0)$$

And

$$m \left. \frac{dG}{dt} \right|_{t = t_0 + \varepsilon} = mC = 1 \Rightarrow C = \frac{1}{m}$$

So

$$G(t, t_0) = \frac{1}{m} (t - t_0)\, e^{-\eta/2 (t - t_0)}\, \theta(t - t_0)$$

c) The EOM is

$$\left[m \frac{d^2}{dt^2} + m\eta \frac{d}{dt} + m\omega_0^2 \right] G(t, t_0) = \delta(t - t_0)$$

Fourier transforming w.r.t. $\Delta t = t - t_0$

$$G(t - t_0) = \int \frac{d\omega}{2\pi} \, e^{-i\omega(t - t_0)}$$

we find

$$\int_\omega e^{-i\omega\Delta t} \left[-m\omega^2 + m\eta(-i\omega) + m\omega_0^2 \right] G(\omega) = \int \frac{d\omega}{2\pi} e^{-i\omega\Delta t} \, 1$$

Or

$$G(\omega) = \frac{1}{\left[-m\omega^2 + m\eta(-i\omega) + m\omega_0^2 \right]}$$

ω_1 and ω_2 are the roots of this polynom.

i.e.

$$G(\omega) = -\frac{1}{m} \frac{1}{(\omega - \omega_1)(\omega - \omega_2)}$$

$$\omega_1 \equiv \omega_+ = -\frac{i\eta}{2} + \sqrt{\omega_0^2 - (\eta/2)^2}$$

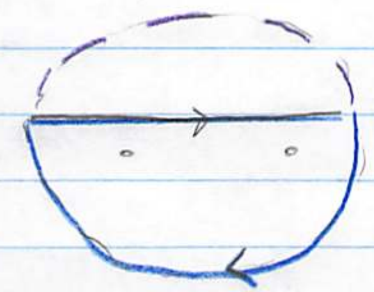
$$\omega_2 \equiv \omega_- = -\frac{i\eta}{2} - \sqrt{\omega_0^2 - (\eta/2)^2}$$

yielding

$$G(t-t_0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t_0)}}{-m(\omega-\omega_1)(\omega-\omega_2)}$$

Where $\omega_1 = \omega_+ \equiv -i\frac{\gamma}{2} + \sqrt{\omega_0^2 - (\gamma/2)^2}$ and $\omega_2 \equiv \omega_-$

d) Now we will do the integral via complex analysis



$t-t_0<0$

$t-t_0>0$

- For $t-t_0<0$ we close above and get zero.
- For $t-t_0>0$ we close below and pickup the poles at $\omega=\omega_1$ and $\omega=\omega_2$, going the "wrong" way around them

$$G(t-t_0) = -2\pi i \left[\operatorname*{Res}_{\omega=\omega_+} f + \operatorname*{Res}_{\omega=\omega_-} f \right] \Theta(t-t_0)$$

where we define:

$$f \equiv \frac{-1}{2\pi m} \frac{e^{-i\omega(t-t_0)}}{(\omega-\omega_+)(\omega-\omega_-)} \qquad \omega_\pm \equiv -i\frac{\gamma}{2} \pm \omega_*$$

$$\omega_* \equiv \sqrt{\omega_0^2 - (\gamma/2)^2}$$

$$\operatorname*{Res}_{\omega=\omega_+} f = \frac{-1}{2\pi m} \frac{e^{-i\omega_+(t-t_0)}}{2\omega_*} = \frac{e^{-\eta/2(t-t_0)}\, e^{-i\omega_*(t-t_0)}}{4\pi m \omega_*}$$

$$\operatorname*{Res}_{\omega=\omega_-} f = +\frac{1}{2\pi m} \frac{e^{-i\omega_-(t-t_0)}}{2\omega_*} = \frac{e^{-\eta/2(t-t_0)}\, e^{+i\omega_*(t-t_0)}}{4\pi m \omega_*}$$

So

$$G(t-t) = \frac{1}{m}\left[\frac{e^{i\omega_*(t-t_0)} - e^{-i\omega(t-t_0)}}{2\omega_* i}\right] e^{-\eta/2(t-t_0)}\, \Theta(t-t_0)$$

$$G(t-t_0) = \frac{1}{m}\left(\frac{\sin\omega_*(t-t_0)}{\omega_*}\right) e^{-\eta/2(t-t_0)}\, \Theta(t-t_0)$$

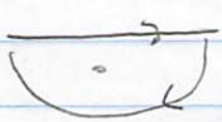
e) In the that $\omega_* \to 0$ then the Green fcn of the previous item becomes

$$G(t-t_0) = \frac{(t-t_0)}{m} e^{-\eta/2(t-t_0)}\, \Theta(t-t_0)$$

in agreement with (b). We can also proceed from the Fourier integral

$$G(t-t_0) = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t_0)}}{-m(\omega-\omega_1)^2}$$

Here there is only a double pole at $\omega_1 = \eta/2$



$$G(t-t_0) = \left[\frac{-2\pi i}{-m} \operatorname*{Res}_{\omega=\omega_1} \frac{e^{-i\omega(t-t_0)}}{(\omega-\omega_1)^2}\right] \Theta(t-t_0)$$

gives double pole   gives single pole

$$e^{-i(\omega_1 + (\omega - \omega_1))(t - t_1)} = e^{-i\omega_1(t-t_0)} + -i(\omega - \omega_1)(t-t_0)e^{-i\omega_1(t-t_0)}$$

$$+ \mathcal{O}((\omega-\omega_1)^2)$$

Then

$$G(t-t') = \frac{-2\pi i}{-m}(-i(t-t_0))\, e^{-i\omega_1(t-t_0)}$$

$$= \frac{(t-t_0)}{m} e^{-\gamma/2 (t-t_0)}$$

Which agrees with before.

Classify

a) Airy $y'' = xy$

- There are no sing points for finite x.
- To analyze, $x \to \infty$ we define $\omega = 1/x$ and study the equation at $w=0$

We note

$$\frac{d}{dx} = -w^2 \frac{d}{dw}$$

$$\frac{d^2}{dx^2} = + w^2 \frac{d}{dw} w^2 \frac{d}{dw} = w^4 \frac{d}{dw} + 2w^3 \frac{d}{dw}$$

So Airy becomes:

$$\left[\frac{d^2}{dw^2} + \frac{2}{w}\frac{d}{dw} - \frac{1}{w^5}\right] y = 0$$

So $w=0$ is an essential singularity. To analyze, substitute e^S. Find

$$(S')^2 = x$$

$$S' = \pm\sqrt{x}$$

$$S = \pm\frac{2}{3} x^{3/2}$$

So the general solution of the Airy Eqn is as $x \longrightarrow \infty$

$$y(x) = C_1 e^{2/3 x^{3/2}} + C_2 e^{-2/3 x^{3/2}}$$

b) For the hypergeometric equation

$$x(1-x)\,y'' + (c-(b+a+1)x)\,y' - ab\,y = 0$$

divide by $x(1-x)$

$$y'' + \frac{(c-(a+b+1)x)}{x(1-x)}\,y' - \frac{ab}{x(1-x)}\,y = 0$$

- This has regular sing points at $x=0$ and $x=1$

$x=0$

Substitute x^s into the most singular part of the eqn:

$$y'' + \frac{c}{x}y' = 0$$

Find $s(s-1)+cs=0$, or $s=0$ and $s=1-c$

$$y = C_1 + C_2\,x^{1-c}$$

$x=1$

for $x=1$ we substitut $(1-x)^s$ into the most singular part of the eqn:

$$y'' + \frac{c-(a+b+1)}{(1-x)}\,y' = 0$$

yielding

$$s(s-1) + -(c-(a+b+1))s = 0$$

$$s(s-\cancel{1}-c+a+b+\cancel{1}) = 0$$

$$s = 0 \quad \text{and} \quad s = c-a-b$$

And thus for x near 1 we have

$$y = C_1 + C_2(1-x)^{c-a-b}$$

For $x \to \infty$ we analyze by subs $x = 1/w$ with w small or x large where the eqn becomes

$$\left[-x^2 \frac{d^2}{dx^2} - (a+b+1)x\frac{d}{dx} - ab \right] y = 0$$

becoming

$$\left[-\frac{1}{w^2}\left(w^4 \frac{d^2}{dw^2} + 2w^3\frac{d}{dw} \right) + (a+b+1)\, w\frac{d}{dw} - ab \right] y = 0$$

i.e

$$\frac{d^2}{dw^2} - \frac{(a+b-1)}{w}\frac{d}{dw} + \frac{ab}{w^2} \Big] y$$

which has a reg sing point as $w \to 0$. Subs w^s and find

$$s(s-1) + (1-a-b)s + ab = 0$$

Or $s = a$ or $s = b$. The general solution (recalling that $w = 1/x$) is

$$y = C_1 \frac{1}{x^a} + C_2 \frac{1}{x^b} \qquad \text{for } x \to \infty$$

c) For Kummer's hypergeometric eqn

$$x y'' + (b-x) y' - a y = 0$$

$$y'' + \frac{(b-x)}{x} y' - \frac{a}{x} y = 0$$

- We have a reg sing point at $x \to 0$
Subs x^s:

$$s(s-1) + bs = 0 \Rightarrow s(s-1+b)$$

$$s = 0, \text{ and } s = 1-b$$

i.e.

$y = C_1 + C_2 x^{1-b}$ is the general solution

- Turning to infinity, $w = 1/x$, and we find

$$\left[\frac{1}{w} \left(w^4 \frac{d^2}{dw^2} + 2 w^3 \frac{d}{dw} \right) + \frac{1}{w} w^2 \frac{d}{dw} - a \right] y = 0$$

Dividing by w^3:

$$\left[\frac{d^2}{dw^2} + \left(\frac{2}{w} + \frac{1}{w^2} \right) \frac{d}{dw} - \frac{a}{w^3} \right] y = 0$$

Which has an irreg sing point at $w=0$

- Substituting $y = e^S$ to analyze it as $x \to \infty$:

neglect

$$x\left(e^S\left(\cancel{S''} + (S')^2\right)\right)$$

$$+ (b-x)e^S S' - a e^S = 0$$

Find

$$x(S')^2 + (b-x)S' - a = 0$$

exact. now take $x \to \infty$

$$S' = \frac{-b+x}{2x} \pm \frac{\sqrt{b^2+4ax-2bx+x^2}}{2x}$$

Or

- $S'_- \simeq -\frac{a}{x}$ (negative root for $x \to \infty$)

- $S'_+ = 1 + \frac{a-b}{x}$ (positive root, $x \to \infty$)

So integrating we find

- $S_- = -a\ln x + \text{const}$

- $S_+ = x + (a-b)\ln x + \text{const}$

So, $y = e^S$ and the general solution is

$$\boxed{y = \underbrace{C_1 x^{-a}}_{e^{S_-}} + \underbrace{C_2 e^x x^{a-b}}_{e^{S_+}} \quad \text{for } x \to \infty}$$

Near a reg point

$$\underbrace{(x-1)(x-2)}^{x^2-3x+2} y'' + (4x-6)y' + 2y = 0$$

- Substitute

$$\bar{y} = \sum_n a_n x^n \qquad a_0 = 1, \; a_1 = 1$$

In particular take $y = \sum a_n x^n$

① $x^2 y'' = \sum n(n-1) a_n x^n$

② $-3x y'' = \sum_n n(n-1) a_n (-3) x^{n-1} = \sum_n x^n (n+1)(n)(-3a_{n+1})$

③ $2y'' = \sum 2n(n-1) a_n x^{n-2} = \sum_n x^n (n+2)(n+1)(2a_{n+2})$

④ $4xy' = \sum_1 4n a_n x^n = \sum_n x^n (4n a_n)$

⑤ $-6y' = \sum_1 -6n a_n x^{n-1} = \sum_n x^n (-6(n+1) a_{n+1})$

⑥ $2y = \sum_0 2a_n x^n = \sum x^n (2a_n)$

- Comparing x^n

$$\Big[n(n-1)a_n - 3(n+1)(n)\,a_{n+1} + (n+2)(n+1)\,2a_{n+2}$$

$$+ 4n a_n - 6(n+1)\,a_{n+1} + 2a_n \Big] = 0$$

Yielding

$$-2\,a_{n+2}(n+2)(n+1) = a_n(n+2)(n+1) - 3a_1(1+n)(2+n)$$

$$a_{n+2} = -\frac{a_n + 3a_{n+1}}{2}$$

Starting with $a_0 = 1$ and $a_1 = 1$, find $a_2 = 1$ and

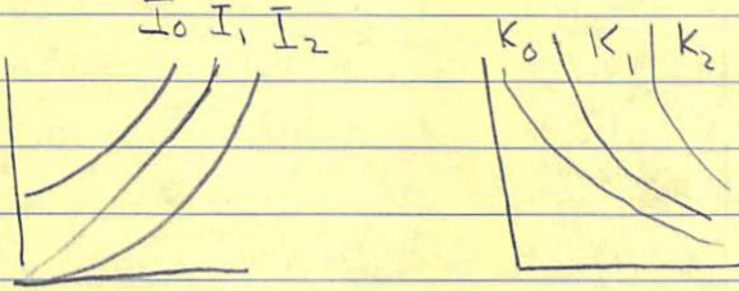
$$a_2 = a_3 \cdots = a_n = 1$$

Thus the solution is

$$y = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

- As expected this series has r.o.c $= 1$. Indeed, $x = 1$ is a singular point of the diff EQ. It is the nearest such point.

a)



b) Then we have the exact equation

$$\left[z\frac{d}{dz} z\frac{d}{dz} - \nu^2 \right] y(z) = z^2 y$$

So $z^2 y$ is a perturbation. The heirarchy is

$$\left[z\frac{d}{dz} z\frac{d}{dz} - \nu^2 \right] y^{(n+2)} = z^2 y^{(n)}$$

c) To find the 0-th solution we substitute $y = z^s$

$$s^2 - \nu^2 = 0 \qquad s = \pm \nu$$

- For $\nu = 2$

$$y = C_1 z^2 + \frac{C_2}{z^2} \qquad z \text{ near } 0.$$

$\boxed{d)}$ We need to solve the heirarchy

$$\left[z \frac{d}{dz} z \frac{d}{dz} - \nu^2 \right] y^{(n+2)} = z^2 \, y^{(n)}$$

- Starting with $y^{(0)} = C\,(z/2)^{\nu}$. Then noting

$$\left[z \frac{d}{dz} z \frac{d}{dz} - 4 \right] y_p(x) = K\, z^a -$$

has particular solution

$$y = \frac{K\, z^a}{a^2 - \nu^2}$$

- We have

$$y^{(2)} = \frac{C\,(z/2)^{\nu}\, z^2}{((2+\nu)^2 - \nu^2)}$$

$$y^{(4)} = \frac{C\,(z/2)^{\nu}\, z^4}{((4+\nu)^2 - \nu^2)((2+\nu)^2 - \nu^2)}$$

$$y^{(6)} = \frac{C\,(z/2)^{\nu}\, z^6}{((6+\nu)^2 - \nu^2)((4+\nu)^2 - \nu^2)((2+\nu)^2 - \nu^2)}$$

- Using $((2k+\nu)^2 - \nu^2) = 4k(k+\nu)$

We have

$$y^{(2k)} = \frac{C\,(z/2)^{\nu+2k}}{k!\,(\nu+k)(\nu+k-1)\dots(\nu+1)}$$

- By redefining $C = \underline{C}\,\frac{1}{\Gamma(\nu+1)}$ we have

$$y^{(0)} = \frac{\underline{C}(z/2)^{\nu}}{\Gamma(\nu+1)}$$

and

$$y^{(2k)} = \frac{\underline{C}\,(z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}$$

Then

$$y_{\nu}(z) = \underline{C}\,(z/2)^{\nu}\sum_{k=0}^{\infty}\frac{(z^2/4)^k}{k!\,\Gamma(k+\nu+1)}$$