Problem 1. Curious Wave Functions

The eigenfunctions of a D7 brane in a curved geometry lead to the following eigenvalue equation of the Sturm Liouville type

$$-\partial_\rho \rho^3 \partial_\rho \phi_n(\rho) = E_n w(\rho) \phi_n(\rho) \qquad w(\rho) = \frac{\rho^3}{(\rho^2+1)^2} \tag{1}$$

where ρ is a radial variable which varies between 0 an ∞.

(a) Determine the general solution for $\phi(\rho)$ near $\rho = 0$ for arbitary values E.

(b) Show that the regular solution at $\rho = 0$ has the series expansion

$$y_1(\rho) = C_1 \left[1 - \frac{E}{8} x^2 + \ldots\right] \tag{2}$$

(c) Show that $\rho \to \infty$ is also a regular singular point and determine the asymptotic form of the general solution solution in this case.

(d) The eigenfunctions $\phi_n(\rho)$ are chosen to satisfy the boundary condtions of regularity at $\rho = 0$ and normalizability at $\rho = \infty$ (i.e. the $\langle \phi, \phi \rangle$ with the inner product below should be finite). With these boundary condtions show that the operator

$$-\frac{1}{w(\rho)} \left[\partial_\rho \rho^3 \partial_\rho\right] \tag{3}$$

is self adjoint on the interval $\rho \in [0, \infty]$ witht the inner product

$$\langle f, g \rangle = \int_0^\infty w(\rho) \phi_n(\rho) \phi_m(\rho) \tag{4}$$

(e) Show that two eigen-functions of different eigenvalues are orthogonal with respect to the inner product.

(f) We have two solutions: $y_1(E, \rho) = \left[1 - \frac{E}{8}\rho^2 + \ldots\right]$ and $y_2(E, \rho) = \frac{1}{\rho^2} + \ldots$ which is normalizable as $\rho \to \infty$. I have numerically integrated y_1 from 0 to some mid point $\rho_{\rm mid}$, and y_2 from $\rho_{\rm max} = 20$ to the mid point $\rho_{\rm mid}$. For strong numerical stability is essential to integrate the solution *away* from the singular points. Download the mathematica notebook that does this, and use it answer the following questions:

(i) Make a graph of $\sqrt{w(\rho)} y_1(\rho)/y_1(\rho_{\rm mid})$ and $\sqrt{w(\rho)} y_2(\rho)/y_2(\rho_{\rm mid})$ for $E = 3$.

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Plot[Evaluate[{Sqrt[w[x]] y1[e, x]/y1[e, xmid],
    Sqrt[w[x]]  y2[e, x]/y2[e, xmid]} /. e -> 3] , {x, xmin, xmax},
 PlotRange -> {{0, 10}, Full }]
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(ii) Plot $y_1(\rho)$ and its series approximation for $\rho = 0 \ldots 1$.

(iii) Plot the wronskian of $y_1(\rho)$ and $y_2(\rho)$ at $\rho = \rho_{\rm mid}$ as a function $E = 5 \ldots 80$. Use it to determine the numerical value of the first three eigenvlaues. Explain your reasoning.

(iv) Plot the first three eigenfunctions (times the weight) the $\sqrt{w(\rho)}\phi_n$.

Problem 2. Damped harmonic oscillator

Take a damped harmonic oscillator on $t \in [a, \infty]$ with a driving force

$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_o^2 \right] y(t) = f(t) \qquad y(a) = y_a \qquad y'(a) = \partial_{t_a} y \tag{5}$$

The associated linear operator is

$$\mathcal{L}_t = \left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_o^2 \right] \qquad y(a) = 0 \qquad y'(a) = 0 \tag{6}$$

and the retarded Green function satisfies with $t_0 \in [a, \infty]$

$$\mathcal{L}_t G(t, t_0) = \delta(t - t_0) \tag{7}$$

(a) Determine the adjoint of the differential equation. Be sure to specify the adjoint boundary conditions.

(b) Show that a formal to the differential equation is

$$y(t) = [\partial_{t_a} y\, G(t, a) - y_a\, \partial_{t_a} G(t, a)] + m\eta G_R(t, a) y_a + \int_a^t dt_0 f(t_0) G(t, t_0) \tag{8}$$

(c) Set the force to zero $f(t)$. Explicitly show that for $t \to t_a$ the solution satisfies the initial conditions, i.e.

$$\lim_{t \to t_a} y(t) = y_a \tag{9}$$

$$\lim_{t \to t_a} \partial_t y(t) = \partial_{t_a} y \tag{10}$$

You will need to establish (by looking at the action of the adjoint) that

$$\lim_{t \to t_o} [m\partial_t \partial_{t_0} G_R(t, t_0) - m\eta \partial_t G_R(t, t_0)] = 0 \tag{11}$$

Problem 3. Particle in a sphere

(a) Consider a particle in a sphere of radius a

$$-\nabla^2\psi = E\psi \qquad \psi(r,\theta,\phi) = \psi \tag{12}$$

Separate variables in spherical coordinates

$$\Psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \tag{13}$$

and find the eignevalue equations in each of the three directions

$$\frac{-1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) + \frac{\ell(\ell+1)}{r^2}R(r) = k^2R^2 \tag{14}$$

$$\left[\frac{1}{\sin\theta}\frac{-\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta + \frac{m^2}{\sin\theta^2}\right]Y_{\ell m} = \ell(\ell+1)Y_{\ell m} \tag{15}$$

$$-\frac{\partial^2\Phi}{\partial\phi^2} = m^2\Phi^2 \tag{16}$$

(b) First examine the third operator in Eq. (16). Show that with periodic boundary conditions the operator is self adjoint.

The eigenfunctions of this equation are

$$\Phi_m(\phi) = e^{im\phi} \qquad m = 0, \pm 1, \pm 2, \ldots \tag{17}$$

(c) Now look at the next differential equation for $\Theta(\theta)$. After a change of variables

$$P(x) \equiv \Theta(\cos\theta) \qquad x \equiv \cos\theta \tag{18}$$

the equation becomes Legendre's differential equation

$$\left[-\frac{d}{dx}(1-x^2)\frac{d}{dx} + \frac{m^2}{1-x^2}\right]P_{\ell m} = \ell(\ell+1)P_{\ell m} \tag{19}$$

Take $m = 0$ for simplicity (azimuthally symmetric functions). The regular singular points of the differential equation are $x = \pm 1, \infty$. The form of the general solution near $x = 1$ is (for ℓ general)

$$y(x) = C_1 + C_2\log(1-x) \tag{20}$$

(The form near $x = -1$ follows from the symmetry $x \to -x$).

Show that (for $m = 0$) if one adopts demands a regular solution at $x = \pm 1$ then the Legendre differential operator with $x \in (-1, 1)$

$$\mathcal{L} \equiv \left[-\frac{d}{dx}(1-x^2)\frac{d}{dx}\right] \qquad y(\pm 1) = \text{regular} \tag{21}$$

is self adjoint.

Physically by requiring regularity at $x = \pm 1$ we are requiring that the wave function fits on the surface of the sphere in much the same way we required that the particle's wave-function "fit" inside the box.

(d) The regular solution near $x = 1$ is known as $P_\ell(x)$. Show that the regular solution near $x = 1$ takes the form

$$P_\ell(x) = 1 + \frac{\ell(\ell+1)}{2}(x-1) + \ldots \tag{22}$$

The symmetry of the equation under $x \to -x$ shows that there is another solution near $x = -1$ of the form

$$P_\ell(-x) = 1 - \frac{\ell(\ell+1)}{2}(1+x) + \ldots \tag{23}$$

(e) We will adjust $\nu = \ell$ so that the wave functions $P_\nu(x)$ and $P_\nu(-x)$ join smoothly at $x = 0$. Physically we are adjusting ν so that the particles wave function fits (gracefully) inside the sphere. From a numerical standpoint one starts with the form in Eq. (22), and integrates the differential equation from $x = 1$ to $x = 0$. One then starts with Eq. (23) and then integrates from $x = -1$ to $x = 0$. For a specific value of ν (ν non-negative integers it turns out) the two solutions $P_\nu(x)$ and $P_\nu(-x)$ will connect smoothly (i.e. their Wronskian will vanish as in problem one). Use mathematica and plot $P_\nu(x)$ and $P_\nu(-x)$. for $\nu = 1.8, 2.0, 2.2$ for $x = -1\ldots 1$,

(f) Now let us analyze this procedure analytically. For analytical work it is easier to start from $x = 0$ where the function is regular and integrate the differential equation towards $x \to \pm 1$. For ℓ integers we expect to find a regular solution.

$x = 0$ is a regular point. Develop a series solution to Eq. (19) which is even in x

$$y_\ell(x) = \sum_n a_n(\ell) x^n \tag{24}$$

with $a_0 = 1$ and $a_1 = 0$.

(i) Determine the coefficients $a_n(\ell)$. This is the series solution of the symmetric combination $P_\ell(x) + P_\ell(-x)$ for generic ℓ.

(ii) On general grounds what is the expected radius of convergence of this series? Explain. Using the explicit form of $a_n(\ell)$ determine the radius of convergence of the series for generic ℓ and show that it matches this expectation.

(iii) Argue that in order to avoid a divergent solution for $x \to 1$ we must have ℓ a non-negative *even* integer. Determine (up to normalization) the second Legendre polynomial $P_2(x)$ from your series solution.

Remark A similar excercise with odd functions $(P_\ell(x) - P_\ell(-x))$ which have $a_0 = 0$ and $a_1 = 1$ (instead of even functions which had $a_0 = 1$ and $a_1 = 0$) leads to legendre polynomials $P_\ell(x)$ with ℓ non-negative *odd* integers.

(g) (Optional) Finally, look at the radial equation In this case the differential equation has a regular singular point at $r = 0$ and an essential singularity at $r \to \infty$. Straightforward analysis near $r = 0$ gives leading behaviour of the general solution

$$R(r) = C_1 r^\ell + \frac{C_2}{r^\ell} \tag{25}$$

The general solution is

$$R(r) = C_1 j_\ell(kr) + C_2 n_\ell(kr) \tag{26}$$

where $j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}(x)$ and $n_\ell(x) = \sqrt{\pi} 2x Y_{\ell+\frac{1}{2}}(x)$ are *spherical bessel functions.* For our purposes we record the behavior at $r = 0$

$$j_\ell(kr) = \frac{r^\ell}{(2n+1)!!} \tag{27}$$

$$n_\ell(kr) = -\frac{r^{-\ell}}{(2n+1)!!} \tag{28}$$

and for $r \to \infty$

$$j_\ell(kr) = \frac{\sin(z - \frac{1}{2}\ell\pi)}{z} \tag{29}$$

$$n_\ell(kr) = -\frac{\cos(z - \frac{1}{2}\ell\pi)}{z} \tag{30}$$

Explain why the eigen functions are

$$R_{n\ell} = C_1 j_\ell(k_{n\ell} r) \tag{31}$$

where

$$k_{n\ell} a = x_{n\ell} \tag{32}$$

are the zeros of $J_{\ell+\frac{1}{2}}(x)$.

(h) Determine the orthorgonality and completeness relations for the spherical bessel functions $j_\ell(kr)$. Explain your reasoning using the general theory of Sturm Liouville operators. You do not need to explicitly evaluate any definite integrals that are needed to normalize your eigenfunctions $j_\ell(kr)$.

Remark: As we know as $a \to \infty$ the eigenfunctions of the laplacian are

$$e^{i\boldsymbol{k}\cdot\mathbf{r}}\,. \tag{33}$$

The relationship between this eigenfunction and the eigenfunctions we have developed here is given by the important expansion

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta) \tag{34}$$