Curious Wave Functions

$\boxed{a)}$ The point $\rho = 0$ is a regular singular point. Substituting ρ^S, we have

$$-s(s+2)\rho^{S+1} = \underbrace{E\rho^{S+3}}_{\text{small for } \rho \to 0}$$

- Thus we find a solution set for $S = 0$ and $s = -2$, i.e.

$$\phi(\rho) = C_1 + \frac{C_2}{\rho^2}$$

$\boxed{b)}$ Then we substitute a series solution

$$\phi = C(1 + a_1\rho + a_2\rho^2 + \ldots)$$

- The equations without approximation take the form

$$-\phi'' - \frac{3}{\rho}\phi' = E\frac{w(\rho)}{\rho^3}\phi(\rho)$$

We notice that $\frac{w(\rho)}{\rho^3} = \frac{1}{(\rho^2+1)^2}$ is even under

$\rho \to -\rho$ so we try a solution in even powers

$$\phi = C_1 (1 + a_2 \rho^2 + \dots)$$

$$= \phi^{(0)} + \phi^{(2)} + \dots$$

- At zeroth order we get a constant $\phi^{(0)} = C_1$

$$-\phi^{(0)\prime\prime} - \frac{3}{\rho} \phi^{(0)\prime} = 0 \qquad \underline{\text{(0-th order)}}$$

- 2nd order yields

$$-\phi^{(2)\prime\prime} - \frac{3}{\rho} \phi^{(2)\prime} = E \cdot 1 \, \phi^{(0)} \qquad \underline{\text{(2nd order)}}$$

where we approximated $\frac{1}{(\rho^2+1)^2} \simeq 1$ at small ρ

- Trying an ansatz $\phi^{(2)} = K\rho^2$ we find

$$-2K - 3 \cdot 2K = E \cdot 1 \cdot C_1$$

or $\quad K = -C_1/8\,E$.

Summarizing

$$\phi = \phi^{(0)} + \phi^{(2)}$$

$$= C_1 \left[1 - \frac{E}{8} \rho^2 + \dots \right]$$

$\boxed{c)}$ For $\rho \to \infty$ we have

$$-\partial_\rho^2 \phi - \frac{3}{\rho}\partial_\rho \phi = E\,\frac{\phi(\rho)}{\rho^4}$$

- Substituting $\frac{1}{\rho^s}$ we find

small as $\rho \to \infty$

$$\left[-s(s+1) + 3s\right]\frac{1}{\rho^{s+2}} = \frac{E}{\rho^{s+4}}$$

- So we find solutions if

$$-s(s-2) = 0$$

i.e. the general solution is

$$\phi = C_1 + \frac{C_2}{\rho^2}$$

$\boxed{d}$

- Now consider

$$I \equiv \langle \mathcal{L} f, g \rangle$$

where

$$\mathcal{L} = -\frac{1}{w(\rho)} \partial_\rho \rho^3 \partial_\rho$$

Then

$$I = \int_0^\infty w(\rho) \left(-\frac{1}{w(\rho)} \frac{\partial}{\partial\rho} \rho^3 \frac{\partial}{\partial\rho} f \right)^* g$$

$$= -\rho^3 \frac{\partial f^*}{\partial\rho} g \Big|_0^\infty + \int_0^\infty d\rho\, \rho^3 \frac{\partial f^*}{\partial\rho} \frac{\partial g}{\partial\rho}$$

$$= \underbrace{-\rho^3 \left(\frac{\partial f^*}{\partial\rho} g - f^* \frac{\partial g}{\partial\rho} \right)\Big|_0^\infty}_{} + \underbrace{\int_0^\infty d\rho + w(\rho) f^* \left[-\frac{1}{w(\rho)} \frac{\partial}{\partial\rho} \rho^3 \frac{\partial}{\partial\rho} g \right]}_{= \langle f, \mathcal{L} g \rangle}$$

- These vanish if f and g behave as constant as $\rho \to 0$, and $1/\rho^2$ as $\rho \to \infty$

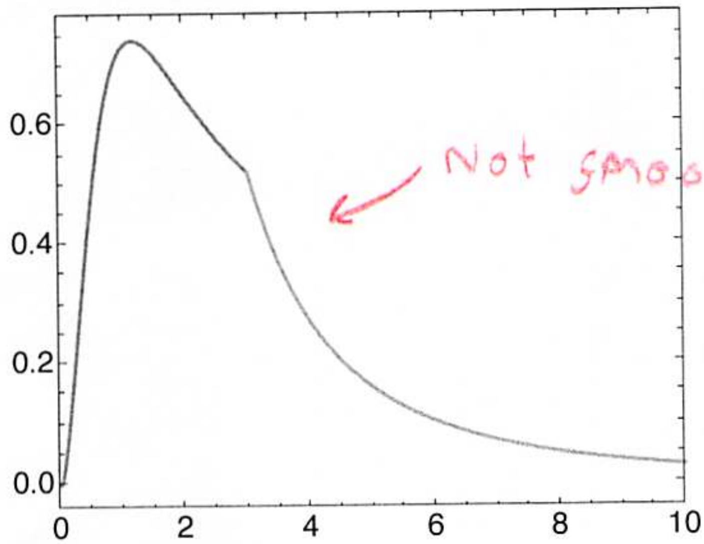
what we want

So

$$\langle \mathcal{L} f, g \rangle = \langle f, \mathcal{L} g \rangle$$ the operator is self adjoint

For general E the solution is not continuous

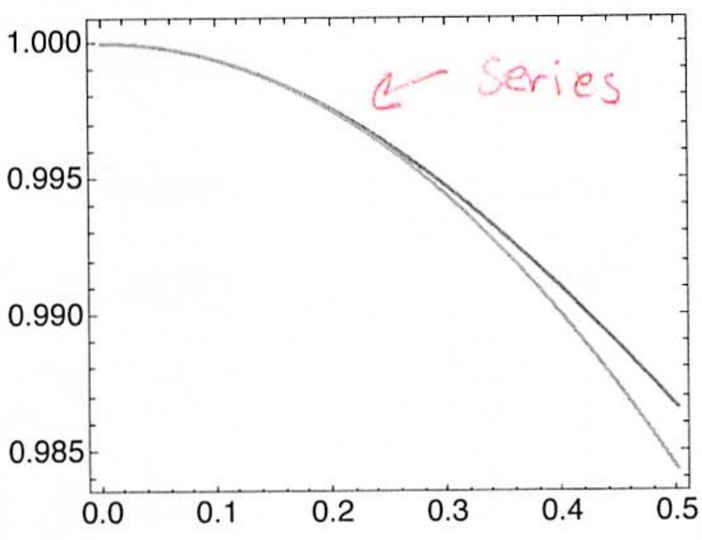
```
Plot[Evaluate[
  {Sqrt[w[x]] y1[e, x] / y1[e, xmid], Sqrt[w[x]] y2[e, x] / y2[e, xmid]} /. e → 3],
 {x, xmin, xmax}, PlotRange → {{0, 10}, Full }]
```



Not smooth for general E

Check the series expansion

```
Plot[{y1[0.5, x], y1small[0.5, x]}, {x, 0, 0.5} ]
```

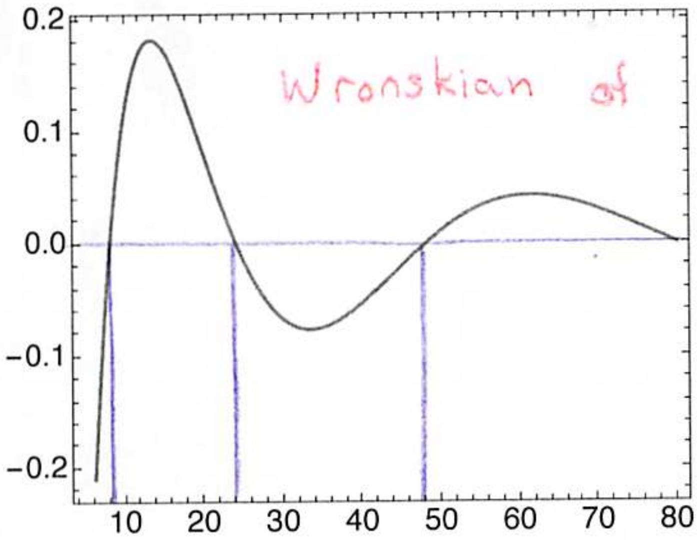


Series Solution follows numerics

Now we plot the wronskian

The eigenvalues are 8, 24, 48 which are the zeros of the wronskian

```
Plot[W[e], {e, 5, 80}]
```



Wronskian of Two solutions

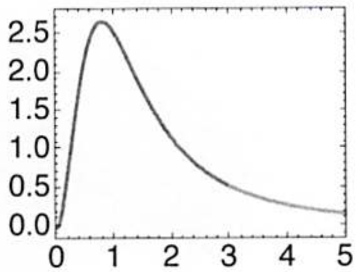
8 24 48

```
Table[4 (n + 1) (n + 2), {n, 0, 2}]
```

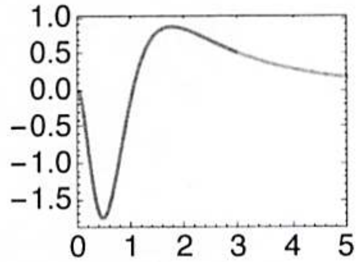
{8, 24, 48}

Now we show the first three eigenfunctions

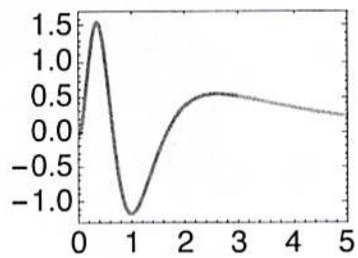
```
Table[Plot[Evaluate[{Sqrt[w[x]] y1[e, x] / y1[e, xmid],
      Sqrt[w[x]] y2[e, x] / y2[e, xmid]} /. e → ee], {x, xmin, xmax},
   PlotRange → {{0, 5}, Full}], {ee, {8, 24, 48}}] // TableForm
```



E = 8



E = 24



E = 48

Harmonic Oscillator

a) Take

$$I = \int_a^\infty dt \; f \mathcal{L} g = \int_a^\infty dt \; f(t) \left[m\frac{d^2}{dt^2} + m\eta \frac{d}{dt} + m\omega_0^2 \right] g$$

• Integrating by parts twice:

$$I = m\left(f \frac{dg}{dt} - g\frac{df}{dt}\right)\Big|_a^b + m\eta f g \Big|_a^b$$

$$+ \int_a^\infty dt \left[\left(m\frac{d^2}{dt^2} - m\eta\frac{d}{dt} + m\omega_0^2 \right) f^* \right] g$$

Then if $\underset{=0}{g(a)}$ and $g'(a) = 0$, then the boundary terms vanish if

$$f(b) = f'(b) = 0$$

So then

$$I = \int_a^b dt \left[\left(m\frac{d^2}{dt^2} - m\eta\frac{d}{dt} + m\omega_0^2 \right) f^* \right] g$$

$$\mathcal{L}^\dagger = \left(m\frac{d^2}{dt^2} - m\eta\frac{d}{dt} + m\omega_0^2 \right) \quad \text{with } y(b) = y'(b) = 0$$

b) Consider

$$y(t) = \int_a^b dt_0 \, \mathcal{L}^{\dagger}_{t_0} G(t,t_0) \; y(t_0)$$

Here we used

$$\mathcal{L}^{\dagger}_{t_0} G(t,t_0) = \delta(t-t_0)$$

And

$$\mathcal{L}^{\dagger}_0 = \left[m\frac{d^2}{dt_0^2} - m\gamma \frac{d}{dt_0} + m\omega_0^2 \right]$$

with b.c. $y(b) = y'(b) = 0$

Integrating by parts twice we find

$$y(t) = m\left(\frac{\partial G(t,t_0)}{\partial t_0} y(t_0) - G(t,t_0) \frac{\partial y(t_0)}{\partial t_0} \right)\Bigg|_a^b$$

$$- m\gamma \, G(t,t_0) \, y(t_0) \Bigg|_a^b + \int_a^b G(t,t_0) \mathcal{L}_{t_0} y(t_0)$$

Using the adjoint boundary conditions that $G(t,t_0)$ satisfies

$$G(t,t_0)\Bigg|_{t_0=b} = \frac{\partial G(t,t_0)}{\partial t_0}\Bigg|_{t_0=b} = 0 \, ,$$

For $l = 4$ we find

$$y = 1 - \frac{4 \cdot 5}{2} x^2 + \frac{(2 \cdot 3 - 4 \cdot 5)(-4 \cdot 5)}{4!} x^4$$

$$= 1 - 10x^2 + \frac{35}{3} x^4$$

Wh

we find finally

$$\boxed{y(t) = -m\left(\frac{\partial G(t,t_0)}{\partial t_0}\,y(t_0) - G(t,t_0)\,\frac{\partial y}{\partial t_0}\right)\Bigg|_{t_0=a} + m\eta\, G(t,t_0)y\Big|_{t_0=a} + \int_a^b dt_0\, G(t,t_0)\, f(t_0)}$$

Where we used

$$\mathcal{L}\, y(t) = f(t) \quad \text{i.e. } m\frac{d^2y}{dt^2} + m\eta\frac{dy}{dt} + m\omega_0^2 y = f$$

c)

Now set $f=0$

$$\boxed{y(t) = m\left[\frac{\partial y}{\partial t_0}\,G(t,t_0) - y(t_0)\,\frac{\partial G(t,t_0)}{\partial t_0}\right]_{t_0=a} + m\eta\, G(t,a)\, y(a)} \qquad (C.1)$$

First lets look at

$$y_{lim} \equiv \lim_{t\to a} y(t)$$ ← want to to show this is y_0

Which equals:

$$y_{lim} = m\frac{\partial y}{\partial t_0}G(a+\varepsilon, a) - m\,y(t_0)\frac{\partial G}{\partial t_0}(a+\varepsilon, a) +$$

$$+ \quad m\eta\, G(a+\varepsilon, a)\; y(a)$$

- Now we know that the green function satisfies

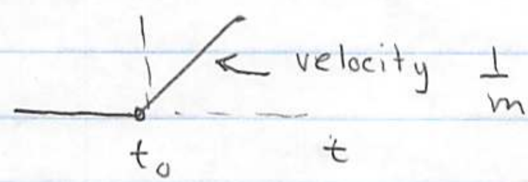
$$\left[m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2 \right] y(t) = \delta(t - t_0)$$

i.e, $=1$

Which describes a particle receiving a unit impulse (kick) at time t_0. Its velocity after the impulse is

$$\frac{\Delta p}{m} = \frac{1}{m} \quad \text{unit}$$

$y(t) = G(t, t_0)$



- So the position is

$$G(t, t_0) = y = \frac{1}{m}(t - t_0) + O((t-t_0)^2)$$

So

$$-m \frac{\partial G(t, t_0)}{\partial t_0} = 1$$

this

From this arguement the three terms above in ☆ are;

$$y_{lim} = O(\varepsilon) + y(t_a)(1) + O(\varepsilon), \text{ i.e}$$

$$y_{lim} = y(t_a) \quad \text{as required. //}$$

• We may also differentiate Eq (C.1)

a couple of pages back

$$\lim_{t \to a} \partial_t y(t) \equiv \dot{y}_{lim}$$

$$\dot{y}_{lim} = m \frac{\partial y}{\partial t_0} \partial_t G(t, t_0) \Big|_{t=t_0+\varepsilon} - m y(t_0) \frac{\partial^2 G}{\partial t \partial t_0} \Big|_{t=t_0+\varepsilon}$$

$$+ m\eta \frac{\partial G}{\partial t} \Big|_{t=t_0+\varepsilon}$$

• Now we procede as before.
After a unit kick the velocity is $\Delta p/m = 1/m$.
It then slows down due to the drag. For short times

$$V(t) = \frac{1}{m} - \frac{\eta}{m}(t - t_0) + O((t-t_0)^2)$$

So the position of the particle up to time t is:

$$G(t, t_0) = \int_{t_0}^{t} V(t') \, dt'$$

$$G(t, t_0) = \frac{1}{m}(t - t_0) - \frac{\eta}{2m}(t-t_0)^2 + O((t-t_0)^3)$$

- Thus we may establish

$$-m\frac{\partial^2 G}{\partial t\,\partial t_0} = -\eta$$

$$m\eta\frac{\partial G}{\partial t} = +\eta$$

$$m\frac{\partial G}{\partial t} = 1$$

- Yielding for the derivative as $t \to t_0$:

$$\dot{y}_{lim} = \left[\frac{\partial y}{\partial t_0}\cdot 1 - \eta\, y(t_0) + \eta\, y(t_0)\right]\Bigg|_{t_0=a}$$

$$\dot{y}_{lim} = \frac{\partial y}{\partial t_0}\Bigg|_{t_0=a}$$

as required.

Partice in a Sphere

(a) Write $\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

- Then the laplacian (times r^2) reads

$$-r^2\nabla^2 = -\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \underbrace{\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{-1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}_{\equiv L^2}$$

The second term does not depend on r. Thus

$$\frac{-r^2\nabla^2\psi}{\psi} = -\frac{1}{R}\frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \frac{1}{Y}L^2 Y = r^2 E \quad k^2$$

- Now at fixed r we can vary θ,ϕ. $\frac{1}{Y}L^2Y$ can not change since r is fixed.

$$\frac{1}{Y}L^2Y = \text{Const}$$

This constant is called $\ell(\ell+1)$. So

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{-\partial^2}{\sin^2\theta\,\partial\phi^2}\right] Y = \ell(\ell+1)\,Y$$

Similarly

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\ell(\ell+1)}{r^2}\right] R(r) = E\,R(r)$$

The Y eqn clearly separates

$$Y = \Theta(\theta)\,\Phi(\phi)$$

- With this form we find

$$-\frac{\partial^2 \Phi}{\partial \phi^2} = m^2 \Phi$$

- and

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{m^2}{\sin^2\theta} \right]\Theta = \ell(\ell+1)\,\Theta$$

$\boxed{b)}$ Consider $\langle f, \mathcal{L} g \rangle$

$$\int_0^{2\pi} d\phi \; f^* \left(-\frac{\partial^2}{\partial\phi^2} g(\phi) \right)$$

$$= \underbrace{-f g' + f' g \Big|_0^{2\pi}}_{\text{bndry terms}} + \int_0^{2\pi} d\phi \left(-\frac{\partial^2}{\partial\phi^2} f \right)^* g$$

Now if f and g are periodic then boundary terms vanish and we have

$$\langle f, \mathcal{L} g \rangle = \int_0^{2\pi} d\phi \left(\frac{-\partial^2 f}{\partial \phi^2} \right)^* g = \langle \mathcal{L} f, g \rangle$$

i.e.

$$\mathcal{L}^\dagger = \mathcal{L}$$

Then

$\boxed{c)}$ $\quad -\dfrac{d}{dx}(1-x^2)\dfrac{d}{dx} = \mathcal{L}$

- We need to show that

$$\langle f, \mathcal{L} g\rangle = \langle \mathcal{L} f, g\rangle \quad \text{where} \quad \langle f, g\rangle = \int_{-1}^{1} dx\, f^* g$$

and f and g are regular at $x = \pm 1$. Then

$$\langle f, \mathcal{L} g\rangle = \int_{-1}^{1} dx\, f^* \left(-\frac{d}{dx}(1-x^2)\frac{d g}{dx}\right) \quad \text{by parts twice}$$

$$= \underbrace{-f^*(1-x^2)g' + f^{*\prime}(1-x^2)g}_{\text{bndry terms}}\Big|_{-1}^{1}$$

$$+ \int_{-1}^{1} dx \left(-\frac{d}{dx}(1-x^2)\frac{d f}{dx}\right)^* g$$

- So if f and g are regular (so that the boundary terms vanish). Then

$$\langle f, \mathcal{L} g\rangle = \int_{-1}^{1} dx \left(-\frac{d}{dx}(1-x^2)\frac{d}{dx} f\right)^* g = \langle \mathcal{L} f, g\rangle$$

i.e. $\langle f, \mathcal{L} g\rangle = \langle \mathcal{L} f, g\rangle$ and the operator is self adjoint

Take

$\boxed{d)}$ $$\left[-\frac{d}{dx}(1-x^2)\frac{d}{dx} - \ell(\ell+1)\right]y = 0$$

- We work near $x = 1$, define $u \equiv x - 1$. Then

$$\left[+\frac{d}{du}\,u(2+u)\frac{d}{du} - \ell(\ell+1)\right]y = 0$$

$$\left[2\frac{d}{du}\,u\frac{d}{du} - \ell(\ell+1)\right]y = \frac{d}{du}u^2\frac{d}{du}y$$

- We try a power series solution $y = \sum a_n u^n = a_0(1 + c_1 u + \ldots)$ with $a_0 = 1$. Then we find

$$2a_0 c_1 - \ell(\ell+1)a_0 \simeq O(u)$$

So through constant order (i.e. neglecting terms of order u) we find

$$\boxed{c_1 = \frac{\ell(\ell+1)}{2}}$$

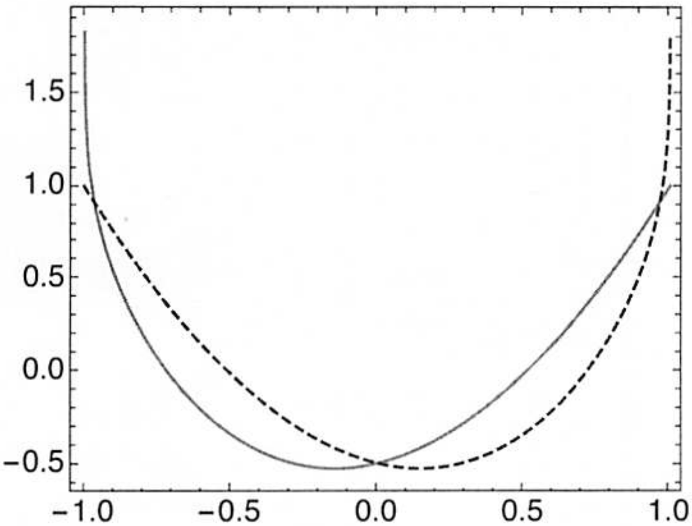
or

$$y(x) = \left(1 + \frac{\ell(\ell+1)}{2}(x-1) + O((1-x)^2) + \ldots\right)$$

Particle in a Sphere Part (e)

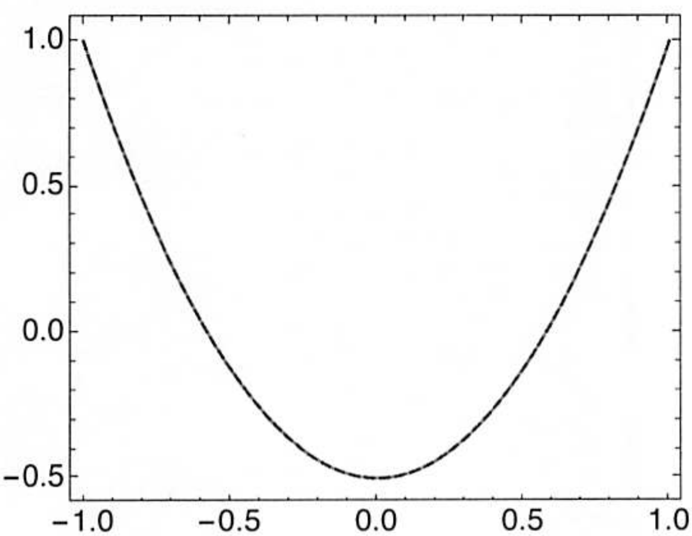
```
Plot[ {LegendreP[1.8, x], LegendreP[1.8, - x] },
 {x, -1, 1} , PlotStyle → {Red, {Blue, Dashed}}]
```

Out[62]=



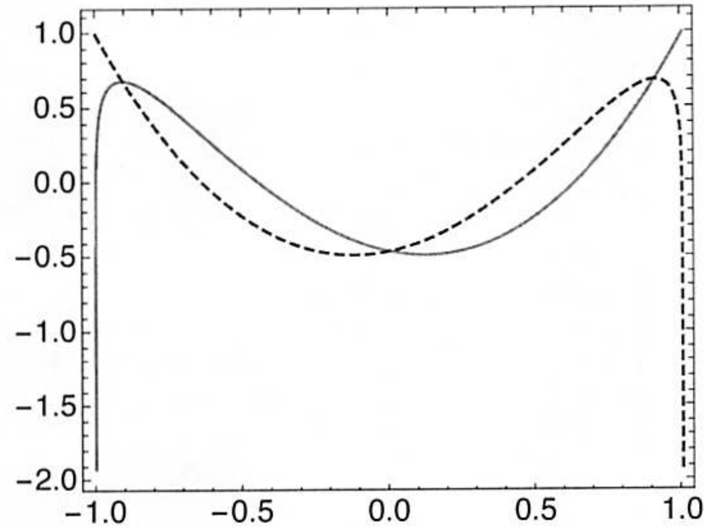
In[64]:=
```
Plot[ {LegendreP[2, x], LegendreP[2, - x] },
 {x, -1, 1}, PlotStyle → {Red, {Blue, Dashed} }]
```

Out[64]=



In[65]:=
```
Plot[ {LegendreP[2.2, x], LegendreP[2.2, - x] },
  {x, -1, 1}, PlotStyle → {Red, {Blue, Dashed} }]
```

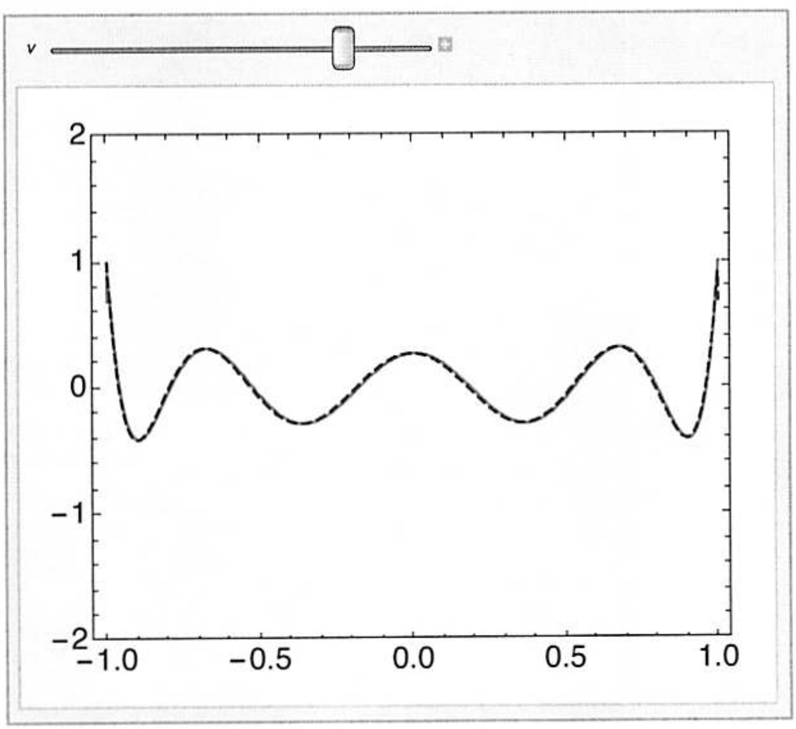
Out[65]=



In[92]:=
```
nlist = Linspace[-0.8, 10.2, (10.2 + 0.8) / (.025) + 1 ]
```

In[113]:=
```
Manipulate[ Plot[ {LegendreP[ν, x] ,  LegendreP[ν, -x]},
   {x, -1, 1}, PlotStyle → {Red, {Blue, Dashed} }, PlotRange → { -2, 2}],
 {{ν, -0.8, "ν"}, nlist}, ControlType → Manipulator,
 Appearance → "Open", AppearanceElements → {"StepRightButton"}]
```

Out[113]=



In[114]:=
```
(6 - 20) (-20) / 4!
```

Out[114]= $\frac{35}{3}$

$\boxed{f)}$ From the DEQ

$$y = \Sigma a_n x^n$$

And $\left[-\frac{d}{dx} (1-x^2) \frac{d}{dx} - \ell(\ell+1) \right] y$

$$-\frac{d^2}{dx^2} y = -n(n-1) a_n x^{n-2} \rightarrow -(n+2)(n+1) a_{n+2} x^n$$

$$\frac{d}{dx} x^2 \frac{d}{dx} y = n(n+1) a_n x^n$$

$$-\ell(\ell+1) y = -a_n x^n (\ell(\ell+1)$$

Comparing terms of order x^n we see

$$-a_{n+2}(n+2)(n+1) + (n(n+1) - \ell(\ell+1)) a_n = 0$$

Or

$$\boxed{a_{n+2} = \frac{[n(n+1) - \ell(\ell+1)] a_o}{(n+2)(n+1)}}$$

- $y = 1 + \frac{-\ell(\ell+1)}{2} x^2 + \frac{(2 \cdot 3 - \ell(\ell+1))(-\ell(\ell+1)}{4!} x^4 + \ldots$

ii) We know on general grounds that the series will converge up to the nearest singular point i.e. $x = \pm 1$

In this case the asymptotic term in the series is

$$a_n x^n \xrightarrow[n \to \infty]{} \frac{x^n}{(n+2)(n+1)}$$

which is clearly greater than unity for $x > 1$ and less than unity for $x < 1$. By comparison with the geometric series the radius of convergence is unity

iii) The only way the series will converge is if n could be l. In this case the series expansion will terminate after a finite # of terms.

$$l = 0, 2, 4 \ldots$$

For $l = 2$ we have

$$y = 1 - \frac{2(2+1)}{2} x^2 + 0x^4 + 0x^6 + 0x^8 + 0 \ldots$$

$y = 1 - 3x^2$ which agrees @ $P_2(x)$ up to normalization

h) The spherical bessels obey the sturm liouville eqn:

$$\left[-\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \ell(\ell+1) \right] R = k^2 \underbrace{r^2}_{W(x)} R$$

According to the general theory we have

$$\int_0^\infty \underbrace{r^2}_{W(x)} \bar{j}_\ell(k_n r)\, j_\ell(k_m r)\, dr = C_{\ell n}\, \delta_{nm}$$

Then the completeness relation reads

$$\sum_n \frac{1}{C_{\ell n}} j_\ell(k_n r)\, j_\ell(k_m r') = \frac{1}{r^2}\, \delta(r - r')$$