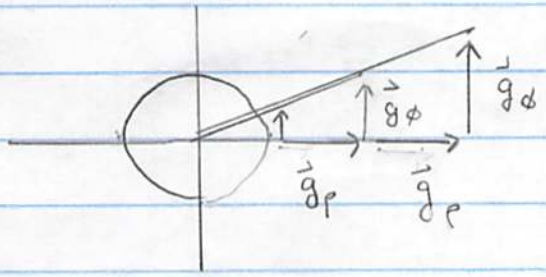
Orthogonal Coordinate Systems

Most of the useful coordinate systems are orthogonal but not orthonormal, e.g. for cylindrical coordinates

$$ds^2 = d\rho^2 + \underbrace{\rho^2}_{\text{Look at me!}} d\phi^2 + dz^2$$



$$\vec{g}_\phi = \rho \sin\phi\, \hat{x} - \rho \cos\phi\, \hat{y}$$

$$\vec{g}_\phi \cdot \vec{g}_\phi = \rho^2$$

length (squared) of $\vec{g}_\phi$ vector

The contravariant vectors are not orthonormal either

$$g^{ab} = \mathrm{diag}\left(1, \frac{1}{\rho^2}, 1\right)$$

$$\vec{g}^{\,\phi} = g^{\phi\phi}\, \vec{g}_\phi = \frac{\sin\phi}{\rho}\, \hat{x} - \frac{\cos\phi}{\rho}\, \hat{y} = \frac{\vec{g}_\phi}{\rho^2}$$

In general an orthogonal coordinate system takes the form

$$ds^2 = h_a^2\, du^a du^a = \underbrace{h_1^2}_{\text{scale factors}} (du^1)^2 + h_2^2 (du^2)^2 + h_3^2 (du^3)^2$$

Define unit vectors

- (Eq 1) $e_{\hat{a}} \equiv \dfrac{\vec{g}_a}{h_a}$

- $\vec{e}^{\hat{a}} \equiv h_a \vec{g}^a$

Note

$h_a = \|\vec{g}_a\|$

Aside:

no sum over a. By convention, the scale factors h_a do not participate in the summation convention, ie,

$h_a du^a$ means $\dfrac{du^1}{u^1}$ or $\dfrac{du^2}{u^2}$ or $\dfrac{du^3}{u^3}$ depending on a. But

$h_a^2 du^a du^a$ is summed over because of $du^a du^a$.

The unit vectors are orthonormal

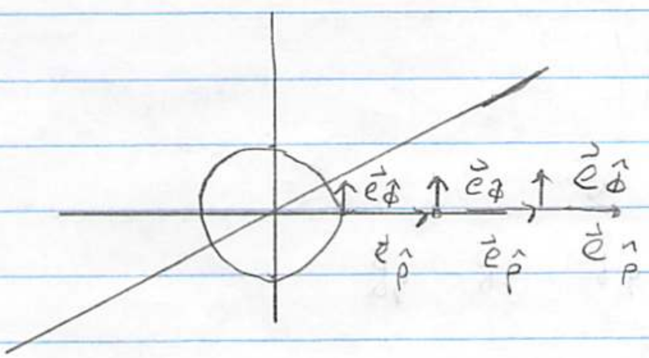
- $\vec{e}_{\hat{a}} \cdot \vec{e}_{\hat{b}} = \dfrac{\vec{g}_a \cdot \vec{g}_b}{h_a h_b} = \dfrac{h_a^2 \, \delta_{\hat{a}\hat{b}}}{h_a^2} = \delta_{\hat{a}\hat{b}}$

b is a because of $\delta_{\hat{a}\hat{b}}$

note no sum over a, as explained above.

For our example

$\vec{e}_{\hat{\rho}} = \hat{\rho} = \vec{g}_\rho$, $\vec{e}_{\hat{\phi}} = \dfrac{g_\phi}{\rho} = \hat{\phi}$, $\vec{e}_{\hat{z}} = \hat{z}$



Then we may expand any vector in the orthonormal basis

$$\vec{V} = V^a \vec{g}_a = (h_a V^a) \frac{\vec{g}_a}{h_a} \equiv \underline{V^{\hat{a}} \vec{e}_{\hat{a}}}$$

Where we have defined (compare to Eq 1):

$$\underline{V^{\hat{a}} \equiv h_a V^a}$$

Similarly find $\vec{V} = V_{\hat{a}} \vec{e}^{\hat{a}}$, with $V_{\hat{a}} = V_a / h_a$ and $\vec{e}^{\hat{a}} = h_a \vec{e}^a$ (compare to Eq. 1 again)

Note that because the vectors are orthonormal:

$$V^{\hat{a}} = h_a V^a = h_a g^{aa} V_a \qquad \text{(no a-sum)}$$

$$= \frac{h_a V_a}{h_a^2} \qquad \text{(no a-sum)}$$

$$V^{\hat{a}} = V_{\hat{a}}$$

This is the "physical" component of the vector $\vec{V}$ along the a-th direction.

Thus there is no need to distinguish between up and down for the hatted vector components

Dot products and cross products work like cartesian coordinates, e.g.

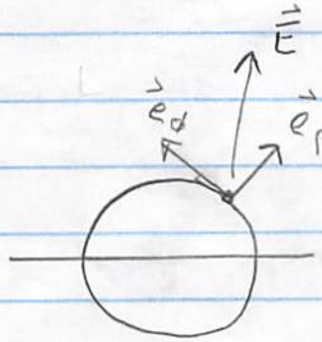
$$\vec{e}_\rho \times \vec{e}_\phi = \vec{e}_z \quad , \quad \vec{e}_z \times \vec{e}_\rho = \vec{e}_\rho$$

Similarly

$$\vec{W}\cdot\vec{V} = W^{\hat{a}} V^{\hat{b}}\, \vec{e}_{\hat{a}}\cdot\vec{e}_{\hat{b}} = W^{\hat{a}} V^{\hat{b}}\, \delta_{\hat{a}\hat{b}}$$

works like cartesian coordinates

Note all books, eg Jackson, Landau etc, except GR books write for the electric field



- $\vec{E} = E^{\rho}\vec{e}_{\rho} + E^{\phi}\vec{e}_{\phi} + E^{z}\vec{e}_{z}$

instead of

- $\vec{E} = E^{\hat{\rho}}\vec{e}_{\rho} + E^{\hat{\phi}}\vec{e}_{\phi} + E^{\hat{z}}\vec{e}_{\hat{z}}$

It is generally clear from context what is meant, i.e. is E^{ϕ} the component of the electric field in the $\hat{\phi}$ direction (and has units V/m), or is E^{ϕ} along $\vec{g}_{\phi}$ in which case E^{ϕ} has units (V/m) 1/m

- Below we list, div, grad curl in a general orthonormal coordinate system.

We will derive these!

Expressions for gradient, divergence, curl etc in orthogonal coordinate systems

Improtant: In this note

$$V^a \text{ is-short-for } V^{\hat{a}} = V^a h_a \qquad\qquad \boldsymbol{e}_a \text{ is-short-for } \boldsymbol{e}_{\hat{a}} = \mathbf{g}_a/h_a \tag{1}$$

as is typical. The metric is

$$ds^2 = (h_a)^2\, \mathrm{d}u^a\, \mathrm{d}u^a = (h_1)^2\,(\mathrm{d}u^1)^2 + (h_2)^2\,(\mathrm{d}u^2)^2 + (h_3)^2\,(\mathrm{d}u^3)^2\,. \tag{2}$$

(a) The volume element is

$$dV = \sqrt{g}\,\mathrm{d}u^1\mathrm{d}u^2\mathrm{d}u^3 = h_1 h_2 h_3\,\mathrm{d}u^1\mathrm{d}u^2\mathrm{d}u^3 \tag{3}$$

(b) The gradient of scalar is

$$\nabla\Phi = \boldsymbol{e}_a \frac{1}{h_a}\frac{\partial\Phi}{\partial u^a} = \frac{\boldsymbol{e}_1}{h_1}\frac{\partial\Phi}{\partial u^1} + \frac{\boldsymbol{e}_2}{h_2}\frac{\partial\Phi}{\partial u^2} + \frac{\boldsymbol{e}_3}{h_3}\frac{\partial\Phi}{\partial u^3} \tag{4}$$

(c) The divergence

$$\nabla\cdot\boldsymbol{V} = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}V^a/h_a)}{\partial u^a} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3V^1)}{\partial u^1} + \frac{\partial(h_1h_3V^2)}{\partial u^2} + \frac{\partial(h_1h_2V^3)}{\partial u^3}\right] \tag{5}$$

(d) The curl

$$(\nabla\times\boldsymbol{V}) = \boldsymbol{e}_a\,[abc]\frac{1}{h_bh_c}\partial_b(h_cV^c) \tag{6}$$

Or

$$\begin{aligned}(\nabla\times\boldsymbol{V}) = \frac{\boldsymbol{e}_1}{h_2h_3}\left[\frac{\partial(h_3V^3)}{\partial u^2} - \frac{\partial(h_2V^2)}{\partial u^3}\right] + \frac{\boldsymbol{e}_2}{h_1h_3}\left[\frac{\partial(h_1V^1)}{\partial u^3} - \frac{\partial(h_3V^3)}{\partial u^2}\right]& \\ + \frac{\boldsymbol{e}_3}{h_1h_2}\left[\frac{\partial(h_2V^2)}{\partial u^1} - \frac{\partial(h_1V^1)}{\partial u^2}\right]&\end{aligned} \tag{7}$$

(e) The Laplacian follows from the divergence and gradient

$$\nabla^2\Phi = \nabla\cdot\nabla\Phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial u^a}\left[\frac{\sqrt{g}}{(h_a)^2}\frac{\partial\Phi}{\partial u^a}\right] \tag{8}$$

$$= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u^1}\left(\frac{\sqrt{g}}{(h_1)^2}\frac{\partial\Phi}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{\sqrt{g}}{(h_2)^2}\frac{\partial\Phi}{\partial u^2}\right) + \frac{\partial}{\partial u^3}\left(\frac{\sqrt{g}}{(h_3)^2}\frac{\partial\Phi}{\partial u^3}\right)\right] \tag{9}$$

Gradient is the simplest to derive

$$\vec{\nabla} f = \frac{\partial f}{\partial x^i}\vec{e}^{\,i}$$

$$= \underbrace{\frac{\partial f}{\partial x^j}\cdot\frac{\partial x^j}{\partial u^a}}\overbrace{\left(\frac{\partial u^a}{\partial x^i}\vec{e}^{\,i}\right)}^{=\vec{g}^{\,a}}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial u^a}\vec{g}^{\,a}$$

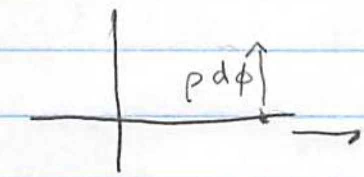
gradient in general coordinates

$$\vec{\nabla} f = \frac{1}{h_a}\frac{\partial f}{\partial u^a}\left(h_a\vec{g}^{\,a}\right)$$

$$= \frac{1}{h_a}\frac{\partial f}{\partial u^a}\vec{e}^{\,a}$$

grad in orthogonal coordinate system

So for example:



$$\vec{\nabla} f = \frac{\partial f}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\vec{e}_\phi$$

gradient in cylidrical coords.

General Coordinate Transformation

- Consider a new coordinate system $\underline{u}^a(u^1, u^2, u^3)$. The jacabian of the transformation is

$$(M)^a{}_b = \frac{\partial \underline{u}^a}{\partial u^b} \qquad (M) = \frac{\partial(\underline{u}^1, \underline{u}^2, \underline{u}^3)}{\partial(u^1, u^2, u^3)}$$

While the inverse transformation has

$$(M^{-1})^a{}_b = \frac{\partial u^a}{\partial \underline{u}^b} \qquad (M^{-1}) = \frac{\partial(u^1 u^2 u^3)}{\partial(\underline{u}^1, \underline{u}^2 \underline{u}^3)}$$

- The differentials du^a transform as M and as a column

$$d\underline{u}^a = \frac{\partial \underline{u}^a}{\partial u^b} du^b = (M)^a{}_b \, du^b$$

- The basis vectors $\vec{g}_a$ transform as a row and with M^{-1}

$$\underline{g}_a = \frac{\partial \vec{s}}{\partial \underline{u}^a} = \frac{\partial \vec{s}}{\partial u^b} \frac{\partial u^b}{\partial \underline{u}^a} = \vec{g}_b \frac{\partial u^b}{\partial \underline{u}^a}$$

$$\vec{\underline{g}}_a = \vec{g}_b \, (M^{-1})^b{}_a$$

- In general the upstairs (contravariant) components transform with $M^a{}_b = \dfrac{\partial \underline{u}^a}{\partial u^b}$, while downstairs (covariant) components transform with M^{-1} and as a row

- Examples:

These rules are often taken as the definition of vectors + tensors

$$V^a \longrightarrow \underline{V}^a(\underline{u}) = M^a{}_b V^b(u)$$

$$V_a \longrightarrow \underline{V}_a(\underline{u}) = V_b(u)(M^{-1})^b{}_a$$

$$T^{ab} \longrightarrow \underline{T}^{ab} = M^a{}_c M^b{}_d T^{cd}$$

$$T^a{}_b \longrightarrow \underline{T}^a{}_b = \underbrace{M^a{}_c T^c{}_d (M^{-1})^d{}_b}_{\text{traditional "similarity" transformation}}$$

- With these rules contractions of indices are always invariant,

$$\vec{V} = V^a \vec{g}_a = \underline{V}^a \underline{\vec{g}}_a$$

$$\overleftrightarrow{T} = T^{ab} \vec{g}_a \vec{g}_b = \underline{T}^{ab} \underline{\vec{g}}_a \underline{\vec{g}}_b$$

physical quantities doesn't change under coordinate transformation

e.g.

$$\underline{\vec{V}} = \underline{\vec{g}}_a \underline{V}^a = \vec{g}_b (M^{-1})^b{}_a (M)^a{}_c V^c = \vec{g}_b \delta^b{}_c V^c$$

$$= \vec{g}_c V^c = \vec{V}$$

Note we have been doing this all along:

- $V^a = \frac{\partial u^a}{\partial x^i} V^i \xrightarrow{\text{proof}} V^a = \vec{g}^{\,a} \cdot \vec{V}$

$$= \frac{\partial u^a}{\partial x^i} e^i \cdot V^j \vec{e}_j$$

- $V_a = \frac{\partial x^i}{\partial u^a} V_i$

$$V^a = \frac{\partial u^a}{\partial x^i} V^j \delta^i_j$$

$$V^a = \frac{\partial u^a}{\partial x^i} V^i$$