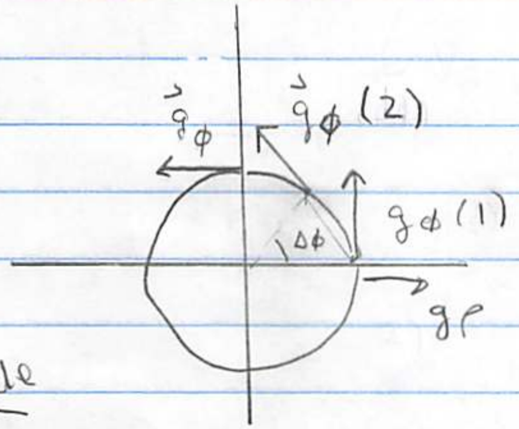
(Cov) Differentiation of Vectors



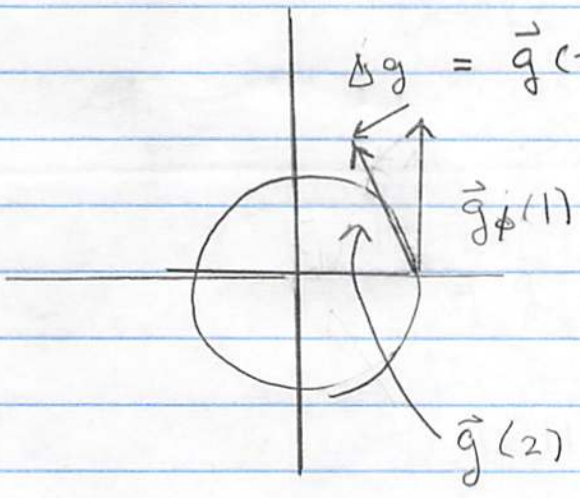
Aside

- Say $V^\phi = 1$

Just because the component is constant doesn't mean that the vector is constant since

$$\vec{V} = V^\phi \vec{g}_\phi = \vec{g}_\phi$$ ← since this is changing

To differentiate a vector we draw a straight line between points 1 and 2. Trace $\vec{g}(2)$ back to 1 (keeping angles fixed) and then difference



In general

$$\frac{\partial \vec{V}}{\partial u^a} = \frac{\partial}{\partial u^a}(V^b \vec{g}_b) = \frac{\partial V^b}{\partial u^a}\vec{g}_b + V^b \frac{\partial \vec{g}_b}{\partial u^a}$$

Then we max expand $\frac{\partial \vec{g}_b}{\partial u^a} \equiv \Gamma^c_{ab} \vec{g}_c$

Thus the derivative of $\vec{V}$ in the u^a direction is

$$\frac{\partial \vec{V}}{\partial u^a} = \left(\frac{\partial}{\partial u^a} V^c + \Gamma^c_{ab} V^b \right) \vec{g}_c \equiv (\nabla_a V^c) \vec{g}_c$$

Here we define:

$$\boxed{\nabla_a V^c \equiv \left(\frac{\partial V^c}{\partial u^a} + \Gamma^c_{ab} V^b \right)} \equiv V^c{}_{;a}$$

covariant derivative

Sometimes we define

$$V^c{}_{,a} \equiv \frac{\partial V^c}{\partial u^a} \qquad V^c{}_{;a} = \nabla_a V^c$$

Derivatives inolve:

these are known as the Christoffel connection

$$\underline{\frac{\partial \vec{g}_b}{\partial u^a} = \Gamma^c_{ab} \vec{g}_c} \qquad \text{or} \qquad \underline{\Gamma^c_{ab} = \vec{g}^c \cdot \frac{\partial \vec{g}_b}{\partial u^a}}$$

We can also express it in terms of $x^i(u^a)$

$$\vec{g}_b = \frac{\partial \vec{s}}{\partial u^b} = \frac{\partial x^i}{\partial u^b} \vec{e}_i \qquad \vec{g}^c = \frac{\partial u^c}{\partial x_i} e_i$$

Thus

$$\underline{\Gamma^c_{ab} = \vec{g}^c \cdot \frac{\partial \vec{g}_b}{\partial u^a} = \frac{\partial^2 x^i}{\partial u^a \partial u^b} \frac{\partial u^c}{\partial x^i}}$$

We can also work out the covariant derivative on lower indices

$$\partial_a (V^b W_b) = (\partial_a V^b) W_b + V^b \partial_a W_b$$

$$= (\partial_a V^b + \Gamma^b_{ac} V^c) W_b + V^b (\partial_a W_b - \Gamma^c_{ab} W_c)$$

Afer relabelling b, c on the 4th of this expression the two added terms are equal
$\Gamma^b_{ac} V^c W_b = \Gamma^c_{ab} V^b W_c$

Thus

$$\partial_a (V^b W_b) = (\nabla_a V^b) \quad + V^b (\nabla_a W_b)$$

where

$$\nabla_a W_b \equiv \partial_a W_b - \Gamma^c_{ab} W_c$$

In general:

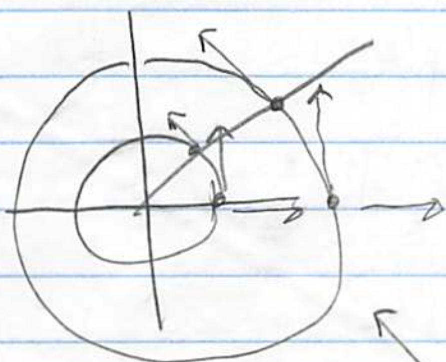
$$\nabla_a T^{c_1 c_2}{}_{d_1 d_2} = \partial_a T^{c_1 c_2}{}_{d_1 d_2} + \Gamma^{c_1}_{ab} T^{b c_2}{}_{d_1 d_2} + \Gamma^{c_2}_{ab} T^{c_1 b}{}_{d_1 d_2}$$

$$- \Gamma^b_{a d_1} T^{c_1 c_2}{}_{b d_2} - T^b{}_{a d_2} T^{c_1 c_2}{}_{d_1 b}$$

Computing Christoffel Symbols:

- $\vec{g}_\rho = \cos\phi \; \hat{x} + \sin\phi \; \hat{y}$
- $\vec{g}_\phi = -\rho\sin\phi \; \hat{x} + \rho\cos\phi \; \hat{y}$

So: $\partial_\rho \vec{g}_\rho = 0$



$$\partial_\rho \vec{g}_\phi = -\sin\phi \; \hat{x} + \cos\phi \; \hat{y} = \frac{\vec{g}_\phi}{\rho}$$

$$\partial_\phi \vec{g}_\rho = \frac{\vec{g}_\phi}{\rho}$$

guarranteed $\partial_\phi \vec{g}_\rho = \partial_\rho \vec{g}_\phi = \partial^2 \vec{s}/\partial\phi\partial\rho$

$\partial_\phi \vec{g}_\phi$ is the acceleration of a particle going round a circle. The acceleration points inward.

$$\left\{ \begin{aligned} \partial_\phi \vec{g}_\phi &= -\rho\cos\phi \; \hat{x} - \rho\sin\phi \; \hat{y} \\ &= -\rho \, \vec{g}_\rho \end{aligned} \right.$$

Now

★ $\partial_a \vec{g}_b = \Gamma^c_{ab} \, \vec{g}_c$

So

$$\Gamma^\rho_{\phi\phi} = -\rho \qquad \Gamma^\phi_{\phi\rho} = \Gamma^\phi_{\rho\phi} = \frac{1}{\rho}$$

just different the basis vector and use Eq. ★ to compute Γ^a_{bc}

Div, grad, curl in general coordinates

① As we will see the covariant derivative

- $\nabla_a V_b$ is tensor

will transform as a tensor. In particular the transformation from cartesian to general coordinates reads

$$\nabla_a V_b = \nabla_i V_j \, (M_o^{-1})^i{}_a (M_o^{-1})^j{}_b, \quad \text{where}$$

$M_o \equiv \partial x^i / \partial u^a$.

② In cartesian coordinates, $\nabla_i = \partial_i$, since the Christoffel connection vanishes in cartesian coordinates, since $\vec{e}_i$ are constant:

Other quantites: $g^{ij} = \delta^{ij}$ in cartesian

Then given an expression for div, grad, curl etc in cartesian coordinates, to find the expression for it in general coords we only need to replace $\partial_i u_j$ with $\nabla_a u_b$ and δ^{ij} with g^{ab}.

Thus . . .

Physical Quantity (Independent of coords)	Cartesian Coords	General Coords
$\nabla \cdot \vec{V}$	$\partial_i V^i$	$\nabla_a V^a$
$\nabla \times \vec{V}$	$\vec{e}_i \, \varepsilon^{ijk} \partial_j V_k$	$\vec{g}_a (\varepsilon^{abc} \nabla_b V_c)$
$\vec{\nabla} f$	$\partial_i f \, \vec{e}^{\,i}$	$\partial_a f \, \vec{g}^{\,a}$
$\nabla^2 f$	$\partial_i \delta^{ij} \partial_j f$	$\nabla_a g^{ab} \nabla_b f$
$\nabla^2 \vec{E}$	$\vec{e}_k \, \partial_i \delta^{ij} \partial_j E^k$	$(\nabla_a g^{ab} \nabla_b E^c) \vec{g}_c$

Example / Proof:

this is δ_{ij} — same as $\nabla_i v_j$ in cartesian

$$\nabla \cdot \vec{V} = \nabla_i V_j \, g^{ij} = \partial_i v_j \, \delta^{ij} \qquad \equiv g^{ab} \text{ by general c.t.}$$

$$= \left(\nabla_i v_j (M_0^{-1})^i{}_a (M_0^{-1})^j{}_b\right)\left((M)^a{}_\ell \, M^b{}_m \, g^{\ell m}\right)$$

this is unity. We have just stuck in $M_0^{-1} M$ for each contracted index

$$\nabla \cdot \vec{V} = (\nabla_a u_b)(g^{ab})$$

In the next section we will show $\nabla_c g^{ab} = 0$ (and similarly $\nabla_c g_{ab} = 0$ etc) yielding

$$\nabla \cdot \vec{v} = \nabla_a (u_b g^{ab}) = \nabla_a V^a$$

Then in cylindircal coords

$$\nabla \cdot \vec{V} = \nabla_a V^a$$

$$= (\partial_a V^a + \Gamma^a_{ab} V^b)$$

$$= (\partial_\rho V^\rho + \partial_\phi V^\phi + \Gamma^\phi_{\phi\rho} V^\rho)$$

We used that the only non-vanishing Christofel symbols are $\Gamma^\rho_{\phi\phi} = -\rho$ and $\Gamma^\phi_{\rho\phi} = \Gamma^\phi_{\phi\rho} = 1/\rho$.

Thus

$$\nabla \cdot \vec{V} = \partial_\rho V^\rho + \partial_\phi V^\phi + \frac{V^\rho}{\rho}$$

← the answer but not quite what is written in book

Use

$$\vec{V} = V^a \vec{g}_a = h_a V^{\hat{a}} \frac{\vec{g}_a}{h_a} = V^{\hat{a}} e_{\hat{a}}$$

← $h_a = \|\vec{g}_a\|$

Thus $V^{\hat{\rho}} = V^\rho$ $\quad V^{\hat{\phi}} = \frac{V^\phi}{\rho}$

So

$$\nabla \cdot \vec{V} = \left(\partial_\rho V^\rho + \frac{V^\rho}{\rho}\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho V^\phi)$$

$$= \left(\partial_\rho V^{\hat{\rho}} + \frac{V^{\hat{\rho}}}{\rho}\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi} V^{\hat{\phi}}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V^{\hat{\rho}}) + \frac{1}{\rho}\frac{\partial}{\partial\phi} V^{\hat{\phi}}$$

Proof that $\nabla_a V_b$ transforms as a tensor

Want to show that

$$\nabla_a V_b = \nabla_i V_j \frac{\partial x^i}{\partial u^a} \frac{\partial x^j}{\partial u^b}$$

First look at why $\partial_a u_b$ is not a tensor

$$\frac{\partial}{\partial u^a} V_b = \frac{\partial}{\partial u^a}\left(\frac{\partial x^j}{\partial u^b} V_j \right)$$

$$= \frac{\partial x^j}{\partial u^b} \frac{\partial V_j}{\partial u^a} + \left(\frac{\partial^2 x^j}{\partial u^a \partial u^b} \right) V_j$$

$$= \frac{\partial x^j}{\partial u^b} \frac{\partial x^i}{\partial u^a} \partial_i V_j + \underbrace{\left(\frac{\partial^2 x^j}{\partial u^a \partial u^b} \right) \frac{\partial u^c}{\partial x^j} V_c}$$

this comes from differentiating the coordinate transformation $\partial x^i / \partial u^b$. Need to subtract off this term. Notice that

$$\frac{\partial^2 x^j}{\partial u^a \partial u^b} \frac{\partial u^c}{\partial x^j} V_c = \Gamma^c_{ab} V_c$$

definition of Christofel from way back

Thus

subtracts off unwanted term

$$\nabla_a V_b = \frac{\partial V_b}{\partial u^a} - \Gamma^c_{ab} V_c$$

$$= \frac{\partial x^j}{\partial u^b} \frac{\partial x^i}{\partial u^a} \underbrace{\partial_i V_j}$$

this is $\nabla_i V_j$ in cartesian coordinate

Formalizing The Covariant Derivative

Notation:

$$\nabla_{\vec{v}} \vec{w} \equiv \text{derivative of } \vec{w} \text{ in direction } \vec{v}$$

① Linearity in $\vec{v}$ and $\vec{w}$

- $\nabla_{a\vec{x}+b\vec{y}} \vec{w} = a\nabla_{\vec{x}} \vec{w} + b\nabla_{\vec{y}} \vec{w}$

② Chain rule

- $\nabla_{\vec{v}} (f\vec{w}) = \underbrace{(\nabla_{\vec{v}} f)}\, \vec{w} + f\,\nabla_{\vec{v}} \vec{w}$

directional derivative of f

$$\vec{v} \cdot \vec{\nabla} f = v^x \partial_x f + v^y \partial_y f$$

With these we have $\qquad = v^a \partial_a f$ in general

$$\nabla_{\vec{v}} \vec{w} = \nabla_{v^a \vec{g}_a} (w^b \vec{g}_b)$$

$$= v^a \partial_a w^b \, \vec{g}_b + v^a w^b \, \nabla_{\vec{g}_a} \vec{g}_b$$

Note we defined:

$$\nabla_a \equiv \nabla_{\vec{g}_a} \qquad\qquad \nabla_a \vec{g}_b = \Gamma^c_{ab} \vec{g}_c$$

definition

Two more conditions (these are more restrictive)

(3) $\partial_a(\vec{v}\cdot\vec{w}) = (\nabla_a \vec{v})\cdot\vec{w} + \vec{v}\cdot\nabla_a\vec{w}$

metric compatibility

(4) Symmetric (Torsion free)

$$\Gamma^c_{ab} = \Gamma^c_{ba} \quad \text{or} \quad \nabla_a \vec{g}_b = \nabla_b \vec{g}_c$$

- These two are clear if there is a local cartesian coordinate system where $\nabla_a \vec{e}_i = 0$ and $\vec{g}_a = \frac{\partial x^i}{\partial u^a}\vec{e}_i$

Then $\nabla_a \vec{g}_b = \underbrace{\frac{\partial^2 x^i}{\partial u^a \partial u^b}}_{\text{Symmetric in } ab} \vec{e}_i$ ← proves (4), uses $\nabla_a \vec{e}_i = 0$

$\nabla_a \vec{v} = \frac{\partial v^i}{\partial u^a}\vec{e}_i$ ← use this to prove (3) uses $\nabla_a \vec{e}_i = 0$

The Christofell Symbols and the metric

- We will now show that given the assumptions of the previous paragraph that

☆ $$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab} \right)$$

Proof:

$$\Gamma^c_{ab} = \vec{g}^{\,c} \cdot \nabla_a \vec{g}_b$$

$$= g^{cd} \underbrace{\vec{g}_d \cdot \nabla_a \vec{g}_b}_{\equiv \Gamma_{(ab)d}} = g^{cd} \vec{g}_d \cdot \vec{\nabla}_b \vec{g}_a$$

switch

$$\Gamma_{(ab)d} = \partial_a (\overbrace{\vec{g}_d \cdot \vec{g}_b}^{g_{db}}) - \vec{g}_b \cdot \nabla_a \vec{g}_d = \partial_a (g_{db}) - \vec{g}_b \cdot \nabla_d \vec{g}_a$$

$$\Gamma_{(ba)d} = \partial_b (\vec{g}_a \cdot g_d) - \vec{g}_a \cdot \nabla_b \vec{g}_d = \partial_b (g_{ad}) - \vec{g}_a \nabla_d \vec{g}_b$$

Adding these two

$$2\Gamma_{(ab)d} = \partial_a (g_{db}) + \partial_b (g_{ad}) - (\overbrace{\vec{g}_b \cdot \nabla_d \vec{g}_a + \vec{g}_a \cdot \nabla_d \vec{g}_b}^{\partial_d (g_{ab})})$$

Thus

$$\Gamma_{(ab)d} = \frac{1}{2} \left(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab} \right)$$

Example

Cylindrical coordinates

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

The only non-trival component is $g_{\phi\phi} = \rho^2$

So inspection of Eq ★ gives that the only non-vanishing components are

$$\Gamma^{\phi}{}_{\rho\phi} = \Gamma^{\phi}{}_{\phi\rho} \quad \text{and} \quad \Gamma^{\rho}{}_{\phi\phi}$$

Differentiating

$$\Gamma^{\phi}{}_{\rho\phi} = \frac{1}{2} g^{\phi\phi} \left(\partial_\rho g_{\phi\phi} + \cancelto{0}{\partial_\phi g_{\phi\rho}} + \cancelto{0}{\partial_\phi g_{\rho\phi}} \right)$$

$$= \frac{1}{2} \frac{1}{\rho^2} \frac{\partial}{\partial\rho} \rho^2 = \frac{1}{\rho} \quad ✓$$

$$\Gamma^{\rho}{}_{\phi\phi} = \frac{1}{2} g^{\rho\rho} \left(\cancelto{0}{\partial_\phi g_{\rho\phi}} + \cancelto{0}{\partial_\phi g_{\phi\rho}} - \partial_\rho g_{\phi\phi} \right)$$

$$\Gamma^{\rho}{}_{\phi\phi} = -\rho \quad ✓$$

These equations agree @ before

Other consequences

- $\nabla_a g^{cd} = \nabla_a g_{cd} = 0$

Thus, $g^{ab} \nabla_a V_b = \nabla_a (g^{ab} V_b) = \nabla_a V^a$, and similarly

$$\partial_a (V^b W^c g_{bc}) = (\nabla_a V^b) W_b + V^b \nabla_a W_b$$

- A useful identity following from Eq ☆ above

$$\Gamma^a_{ab} = \frac{1}{\sqrt{g}} \partial_b \sqrt{g}$$

Then

$$\nabla \cdot \vec{V} = \nabla_a V^a = \partial_a V^a + \Gamma^a_{ab} V^b$$

$$= \partial_a V^a + \frac{1}{\sqrt{g}} (\partial_b \sqrt{g}) V^b$$

- $\nabla \cdot \vec{V} = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} V^a)$

Finally

$$\nabla^2 \vec{\phi} = \nabla \cdot \nabla \phi$$

$$(\nabla \phi)^a = g^{ab} \frac{\partial \phi}{\partial u^b}$$

So

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \partial_a \left(\sqrt{g}\, g^{ab} \partial_b \phi \right)$$

Compare $\qquad ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$

$$\nabla \cdot \vec{V} = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g}\, V^a)$$

$$= \frac{1}{\rho} \partial_\rho (\rho V^\rho) + \frac{1}{\rho} \partial_\phi (\rho V^\phi) + \frac{1}{\rho} \partial_z (\rho V^z)$$

$$= \frac{1}{\rho} \partial_\rho (\rho V^{\hat{\rho}}) + \frac{1}{\rho} \frac{\partial}{\partial \phi} (V^{\hat{\phi}}) + \partial_z V^{\hat{z}}$$

usual form of divergence