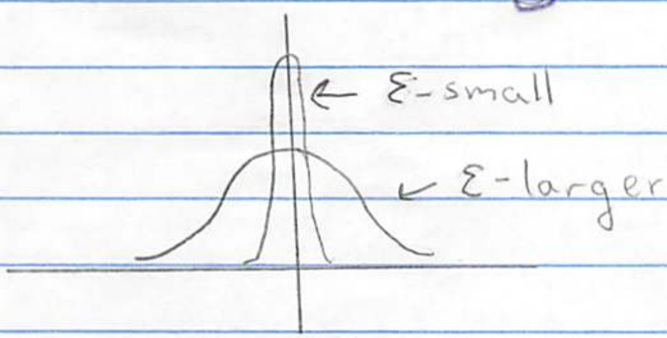
Brief Review of δ-fcns

A Dirac Sequence $\delta_\varepsilon(x)$ a family of (Symmetric $\delta_\varepsilon(-x) = \delta_\varepsilon(x)$) functions parametrized by ε whose integral is unity, but which approaches zero (or more precisely whose integral approaches zero) in any closed interval not containing the origin as $\varepsilon \to 0$

Examples

1. Gaussians (not that common)



$$\frac{1}{\sqrt{2\pi\varepsilon^2}} e^{-x^2/2\varepsilon^2}$$

2. Lorentzians (quite common)

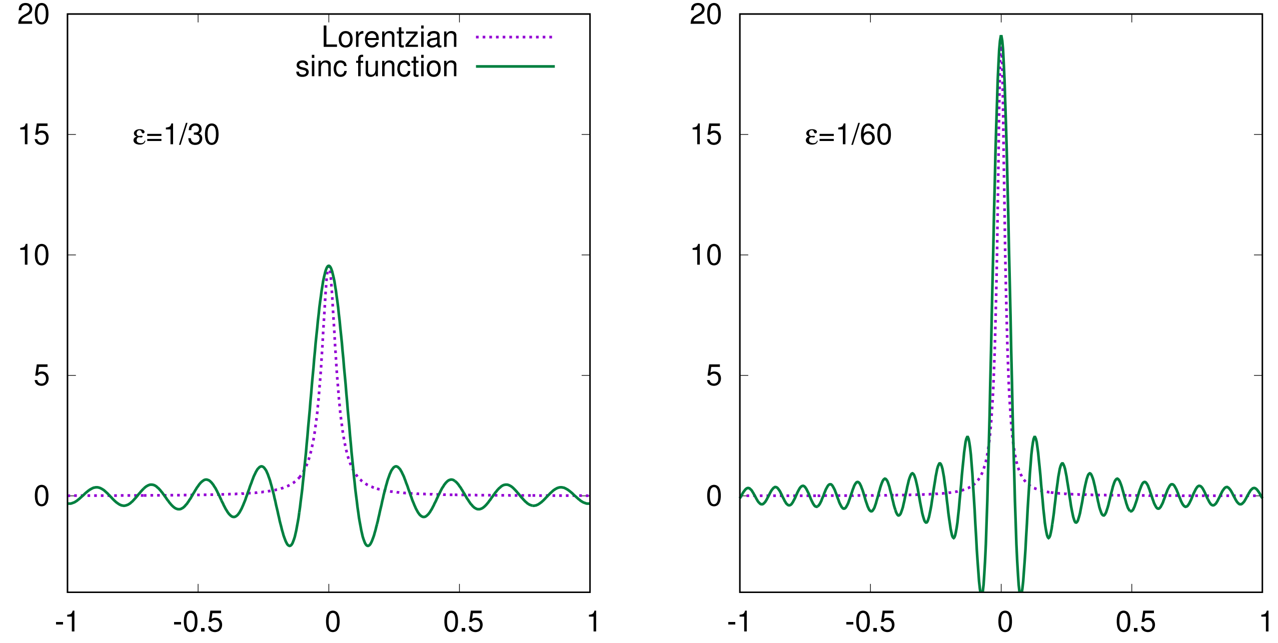
See figure on next page

$$\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$$

3. Sinc function (very common)

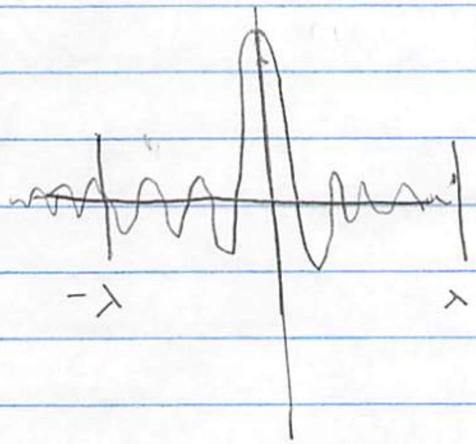
See figure on next page

$$\frac{\sin(x/\varepsilon)}{\pi x}$$



Lorentzian
sinc function
ε=1/30
ε=1/60
20
15
10
5
0
-1
-0.5
0
0.5
1

Lets look at the sinc-fcn (see handout)



- it does not approach zero as $\varepsilon \to 0$ for $x \neq 0$. Only its integral in any interval appraaches 0.

- Pick an interval $[-\lambda, \lambda]$ with $\lambda \ll 1$ but $\lambda \gg \varepsilon$ and integrate

Then

$$I = \int_{-\lambda}^{\lambda} \frac{\sin(x/\varepsilon)}{\pi x}\, dx$$

measuse x in units of ε

$$= \int_{-\lambda}^{\lambda} \frac{\sin(x/\varepsilon)}{\pi (x/\varepsilon)} \frac{dx}{\varepsilon}$$

change variables $u = x/\varepsilon$

$$= \int_{-\lambda/\varepsilon}^{\lambda/\varepsilon} \frac{\sin(u)}{\pi u}\, du$$

Look at limits as $\varepsilon \to 0$ with λ fixed $\lambda/\varepsilon \to \infty$

take $\varepsilon \to 0$

$$= \int_{-\infty}^{\infty} \frac{\sin(u)}{\pi u}\, du$$

$$I = 1$$

We will do this integral Later in course

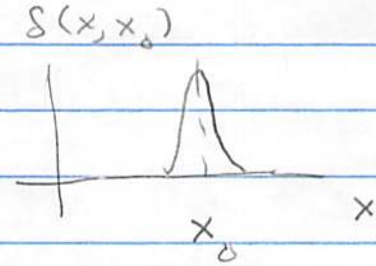
- Thus $\dfrac{\sin(x/\varepsilon)}{\pi x}$ is a dirac sequence!

For any smooth test fcn

$$\lim_{\varepsilon \to 0} \int dx \; \delta_\varepsilon(x) \; f(x) = f(0)$$

Can show: sometimes written $\delta(x, x_0)$

(1) $\int dx \; \delta(x - x_0) \; f(x) = f(x_0)$



(2) $\delta(ax) = \dfrac{1}{|a|} \delta(x)$

(3) $\delta(g(x)) = \sum_a \dfrac{1}{|g'(x_a)|} \delta(x - x_a)$

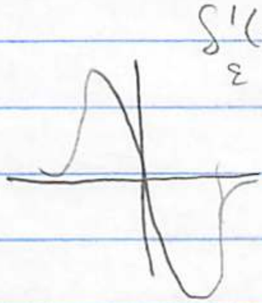
Here x_a are the places where $g(x_a) = 0$ and $g(x) \simeq g'(x_a)(x - x_a)$ near x_a

(4) For any given family of functions can integrate by parts

$$\int_{-x_0}^{x_0} dx \; \frac{d\delta_\varepsilon(x, x_0)}{dx} \; f(x) =$$

$$= \int_{x_0 - \lambda}^{x_0 + \lambda} dx \; \delta(x, x_0) \left(-\frac{df}{dx} \right) = -f'(x_0)$$

But note, since $\delta_\varepsilon(x)$ is even $\delta_\varepsilon(x) = \delta_\varepsilon(-x)$, $\delta'(x)$ is odd:



$$\frac{d}{dx_0}\delta_\varepsilon(x-x_0) = -\frac{d}{dx}\delta_\varepsilon(x-x_0)$$

So

$$\int dx \, \frac{d}{dx_0}\delta(x-x_0)\, f(x) = f'(x_0)$$

(5) Then the integral of the δ-fcn gives

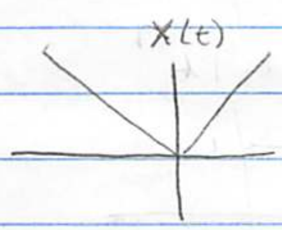
$$\int_{-\infty}^{x} \delta(x-x_0) = \Theta(x-x_0) \quad \text{or } \Theta(x,x_0)$$

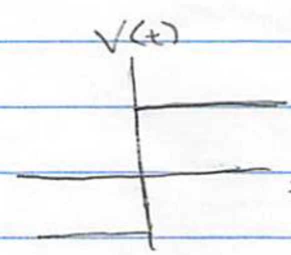
$$= \begin{cases} 1 & \text{if } x > x_0 \\ 0 & x < x_0 \\ \text{ambiguous: usually } 1/2 & \text{if } x = x_0 \end{cases}$$

$$\frac{d}{dx}\Theta(x,x_0) = \delta(x-x_0)$$

Example: Particle with a kick:

$$x(t) = |t|$$





$$a(t) = 2\,\delta(t)$$

Formally

$$x(t) = t\,\Theta(t) - t\,\Theta(-t)$$

$$v(t) = \dot{x} = \left[\Theta(t) + t\,\delta(t) - \Theta(-t) + t\,\delta(t)\right]$$

$$= \left(\Theta(t) - \Theta(-t) + \underline{2t\,\delta(t)}\right)$$

throw away or keep.?

$$a(t) = \left(\delta(t) + \delta(t) + \underline{2\delta(t) + 2t\,\delta'(t)}\right)$$

use (4)

$$= \left(2\delta(t) + \underline{2\delta(t) - 2\delta(t)}\right)$$

$$a(t) = 2\delta(t)$$

doesn't matter

In cases where there is ambiguity, you will need to use fourier analysis. We will turn to this next.

The Hilbert space:

- For definiteness take periodic functions from $[-L/2, L/2]$, $f(x)$, sometimes called vectors.
- Closed under addition and multiplication by complex numbers, etc, i.e.

$$k_1 f_1(x) + k_2 f_2(x)$$ is in the Hilbert Space

- Comes with an inner product

$$\langle g, f \rangle = \text{complex number}$$

which is a linear map from the functions to numbers. Almost always in physics this takes the form:

$$\int_a^b dx \; g^*(x) \, f(x) \; W(x)$$

positive definite weight function. This is 1 for peridic fcns

Required properties:

① $\langle g, k_1 f_1 + k_2 f_2 \rangle = k_1 \langle g, f_1 \rangle + k_2 \langle g, f_2 \rangle$

② $\langle g, f \rangle^* = \langle f, g \rangle$

So

$$\langle k_1 g_1 + k_2 g_2, f \rangle = k_1^* \langle g_1, f \rangle + k_2^* \langle g_2, f \rangle$$

(3) Then $\langle f, f \rangle \geq 0$ with $\langle f, f \rangle = 0$ iff $f = 0$

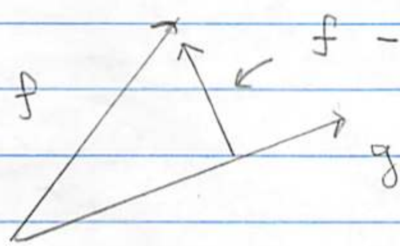
The norm is defined as:

$$\| f \| \equiv \sqrt{\langle f, f \rangle}$$

And $\| f \| = \left(\int dx \, |f|^2 \right)^{1/2}$ is known as the L^2 norm.

The cauchy-schwarz and triangle inequalities

(1) Cauchy Schwarz $|\langle f, g \rangle| \leq \| f \| \, \| g \|$



$$f - \frac{\langle g, f \rangle}{\langle g, g \rangle} g \equiv f_\perp$$

this is the piece of f "orthogonal to g"

Then

$$\langle f_\perp f_\perp \rangle \geq 0$$

Its "length" must be greater than zero.

Then with $\lambda = \langle g,f\rangle / \langle g,g\rangle$ find:

$$\langle f-\lambda g, f-\lambda g\rangle = \langle f,f\rangle - \lambda^* \langle g,f\rangle - \lambda\langle f,g\rangle + |\lambda|^2 \langle g,g\rangle$$

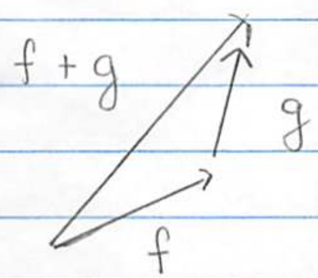
$$= \langle f,f\rangle - \frac{\langle f,g\rangle\langle g,f\rangle}{\langle g,g\rangle} \geqslant 0$$

Or multiplying by $\langle g,g\rangle$ and taking $\sqrt{\ }$:

$$\|f\|\,\|g\| \geqslant |\langle f,g\rangle|$$

(2) Triangle Inequality

$$\|f+g\| \leq \|f\| + \|g\|$$



Proof: Look at $\langle f+g, f+g\rangle$
and use Cauchy Schwarz.
See Book.

Complete Sets of fcns:

Suppose we have a set of functions which are arthogonal

(1) $$\langle u_n, u_m \rangle = \int_a^b w(x)\, u_n^*(x)\, u_m(x) = C_n \delta_{nm}$$

We will find the best approximation of a given function through the expansion

(2) $$f_{approx}(x) = \sum_n f_n \frac{u_n(x)}{C_n}$$

$\leftarrow$ inserted for convenience

Then it is not hard to show that

(3) $$f_m = \langle u_m, f \rangle = \int_a^b w(x)\, u_m^*(x)\, f(x)$$

provides the best approximation of $f(x)$ i.e, minimizes

$$\| f - f_{approx} \|$$

(4) A basis is complete if $\| f - f_{approx} \| \to 0$ as more terms are added.

Then $f(x) = f_{approx}(x)$

In this case

$$f(x) = \sum_n f_n \frac{u_n(x)}{C_n} = f_{approx}(x)$$

$$= \sum_n \int_a^b dx' \, w(x') \, u_n^*(x') f(x') \, \frac{u_n(x)}{C_n}$$

$$f(x) = \int_a^b dx' \, w(x') \left(\underline{\underline{\sum_n \frac{u_n^*(x') \, u_n(x)}{C_n}}} \right) f(x')$$

↳ this needs to be $\frac{\delta(x'-x)}{w(x)}$ so that

$$= \int_a^b dx' \, w(x') \, \frac{\delta(x'-x)}{w(x')} \, f(x')$$

$$= f(x)$$

Thus the requirement for completeness is

$$(4) \quad \sum_n \frac{u_n^*(x') \, u_n(x)}{C_n} = \frac{\delta(x'-x)}{w(x')}$$

for x' and x in $[a, b]$

Periodic Fcns

Take periodic fcns from $[-L/2 \ldots L/2]$
A set of orthogonal functions are

$$\left.\begin{aligned} u_n^s(x) &= \sin(k_n x) \\ u_n^c(x) &= \cos(k_n x) \\ u_0(x) &= 1 \end{aligned}\right\} \quad \text{with} \quad k_n = \frac{2\pi n}{L}, \quad n = 1, 2, \ldots$$

Instead of using these it is better to use

$$u_n(x) = e^{ik_n x} \quad \Big\} \quad \text{with} \quad k_n = \frac{2\pi n}{L}$$

$$n = \ldots -2, -1, 0, 1, 2, \ldots$$

Then the orthogonality and expansion take the form

(1) $$\langle u_m, u_n \rangle = \int_{-L/2}^{L/2} dx \; u_m^*(x)\, u_n(x) = L\, \delta_{nm}$$

prove me!

(2) $$f(x) = \frac{1}{L} \sum_n f_n \; u_n(x)$$

or more explicitly, $f(x) = \frac{1}{L} \sum_n f_n e^{ik_n x}$ The coefficients f_n are given by

(3) $$f_n = \int_{-L/2}^{L/2} u_n^*(x)\, f(x) = \int_{-L/2}^{L/2} e^{-ik_n x}\, f(x)$$

Later we will study completeness

The Power Spectrum and Parsevals Theorem

$$\langle f | f \rangle = \int_{-L/2}^{L/2} dx \, |f|^2$$

Using

$$f(x) = \frac{1}{L} \sum_n f_n \, u_n(x)$$

and

$$\langle u_n, u_m \rangle = L \, \delta_{nm}$$

Find

$$\langle f | f \rangle = \frac{1}{L} \sum_n |f_n|^2$$

In general a plot of $|f_n|^2$ vs. n is known as the power spectrum. If not <u>all</u> f_n are taken then

$$\langle f | f \rangle \geq \frac{1}{L} \sum_{n=-N}^{N} |f_n|^2$$ (Bessels Inequality)

Similar inequalities for all complete sets

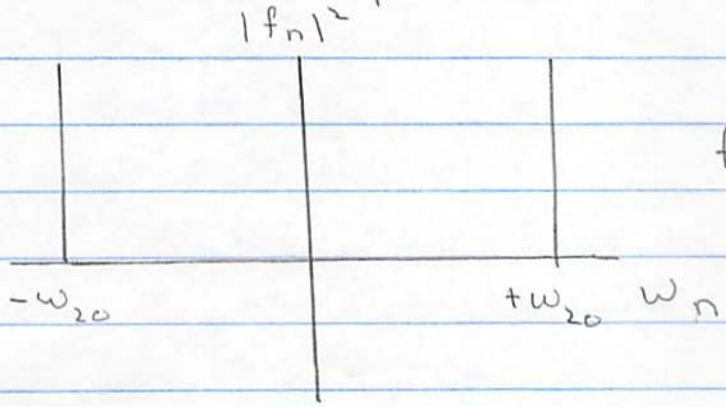
$$\langle f, f \rangle \geq \sum_n \frac{|f_n|^2}{c_n}$$

Example 1: Use time instead of space

$$f(t) = A\cos(\omega_{20} t) \qquad \omega_{20} = \frac{2\pi \cdot 20}{T}$$

$$= \frac{AT}{2}\left[\frac{e^{i\omega_{20}t}}{T} + \frac{e^{-i\omega_{20}t}}{T}\right]$$

Fcns are periodic from $-T/2 \ldots T/2$



$$f_{-20} = f_{20} = \frac{AT}{2}$$

Then

$$\langle f|f\rangle = \int_{-T/2}^{T/2} dt\, A^2 \cos^2 \omega t = A^2 T \langle \cos^2 \rangle = A^2 T/2$$

use $\langle \cos^2 \rangle = \langle \sin^2 \rangle = \frac{1}{2}$ $\quad \langle \cos^2 + \sin^2 \rangle = \langle 1 \rangle = 1$

mean value of $\cos^2$ $\quad \langle \cos\omega t \sin\omega t \rangle = 0$

$$= \frac{1}{T} \sum_n |f_n|^2$$

$$= \frac{1}{T}\left(\frac{A^2T^2}{4} + \frac{A^2T^2}{4}\right)$$

Example 2: AM modulation

See figure on next page

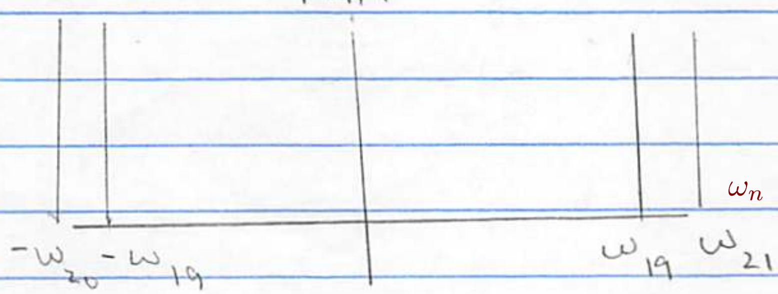
The amplitude depends on time as $A\cos(\omega_1 t)$

$$f(t) = A \underbrace{\cos(\omega_1 t)}_{\text{modulation}} \underbrace{\cos(\omega_{20} t)}_{\text{carrier frequency}} \qquad \omega_n = \frac{2\pi n}{T}$$

$$= \frac{A}{4}\left(e^{i\omega_1 t} + e^{-i\omega_1 t}\right)\left(e^{i\omega_{20} t} + e^{-i\omega_{20} t}\right)$$

$$= \frac{A}{4}\left[\left(e^{i\omega_{21} t} + e^{i\omega_{19} t}\right) + \text{conjugate}\right]$$

Power spectrum



$$f_n = \frac{AT}{4}$$

side bands (non-linear mixing)

Example 3 FM modulation

See figure on next page

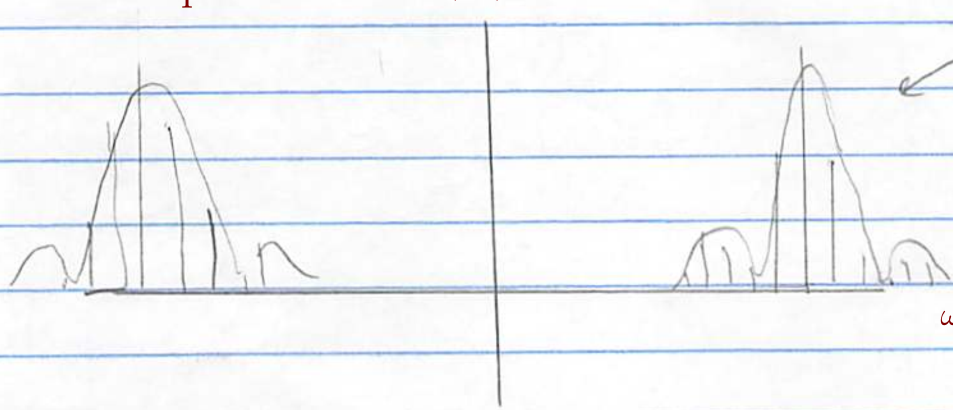
the frequency depends on time

$$f(t) = A\cos(\omega(t)\, t)$$

$$\omega(t) = \omega_{20} - \beta \cos \omega_1 t$$

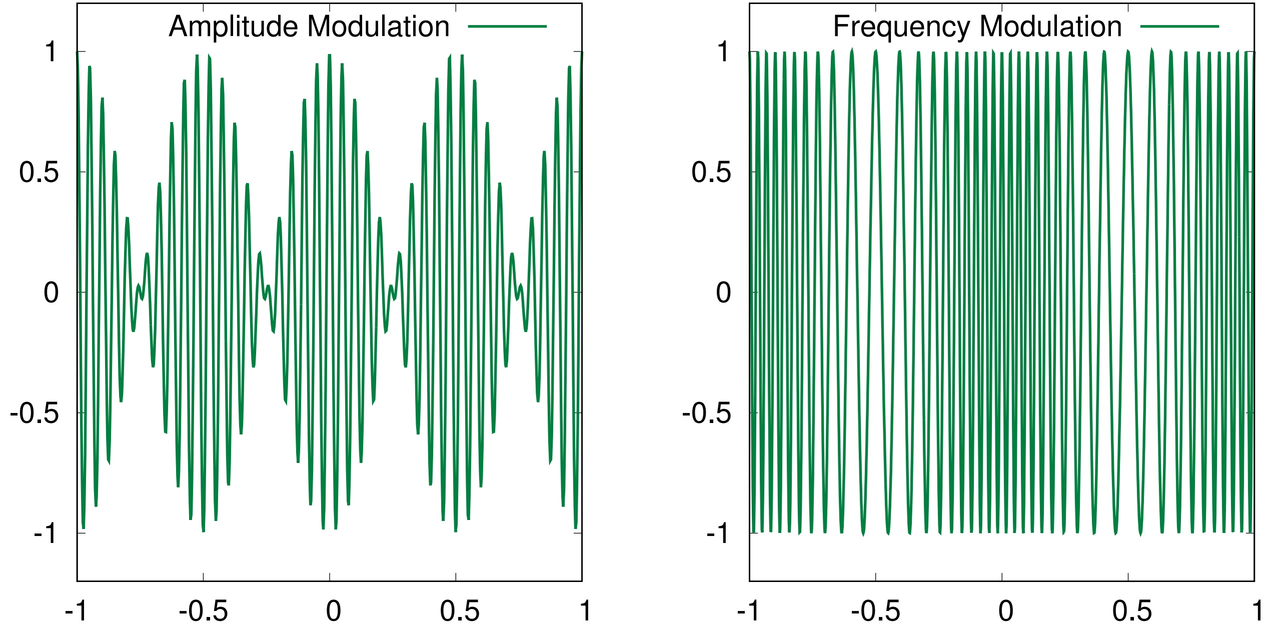
Study in homework

Power spectrum



sidebands described by bessel-fcns

Discussed in homework


Amplitude Modulation
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
Frequency Modulation
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

Completeness of Fourier Series

We must show that for x, y in $[L/_2 \ldots L/_2]$ that

$$\frac{1}{L}\sum_n (e^{ik_n y})(e^{-ik_n x}) = \delta(y-x)$$

More precisely cut off the sum from $-N/2 \ldots N/2$. We should show for x in $(-L; L)$ that

$$\delta_N(x) \equiv \frac{1}{L}\sum_{n=-N/2}^{N/2} e^{ik_n x} \qquad k_n = 2\pi n/L$$

is a dirac sequence as $N \to \infty$ (take $\varepsilon \equiv 1/N$)

This is the most important sum in all of physics! Define $\theta \equiv 2\pi x/L$

$$\delta_N(x) = \frac{1}{L}\sum_{n=-N/2}^{N/2} e^{in\theta} = \frac{1}{L}\sum_{n=-N/2}^{N/2} z^n$$

Use $1 + z + \ldots z^n = (1 - z^{N+1})/(1-z)$ (which you prove by multiplying both sides by $(1-z)$)

$$\delta_N(x) = z^{-N/2} \frac{(1 - z^{n+1})}{(1-z)}$$

$$= \frac{z^{-(N+1)/2} - z^{(n+1)/2}}{z^{-1/2} - z^{1/2}}$$

Take the denominator

$$z^{-1/2} - z^{1/2} = e^{-i\theta/2} - e^{i\theta/2} = 2i\ \sin(\theta/2)$$

Thus

sum of N+1 phases

$$\delta_N(x) = \frac{\sin((N+1)\theta/2)}{\sin(\theta/2)} \qquad \theta = 2\pi x/L$$

this is the diffraction pattern of (N+1) slits. A graph of this function, is given below. See picture below!!

Clearly this function consists of a set of spikes which get narrow and tall. Your homework will be to show that this is a correctly normalized Dirac Sequence

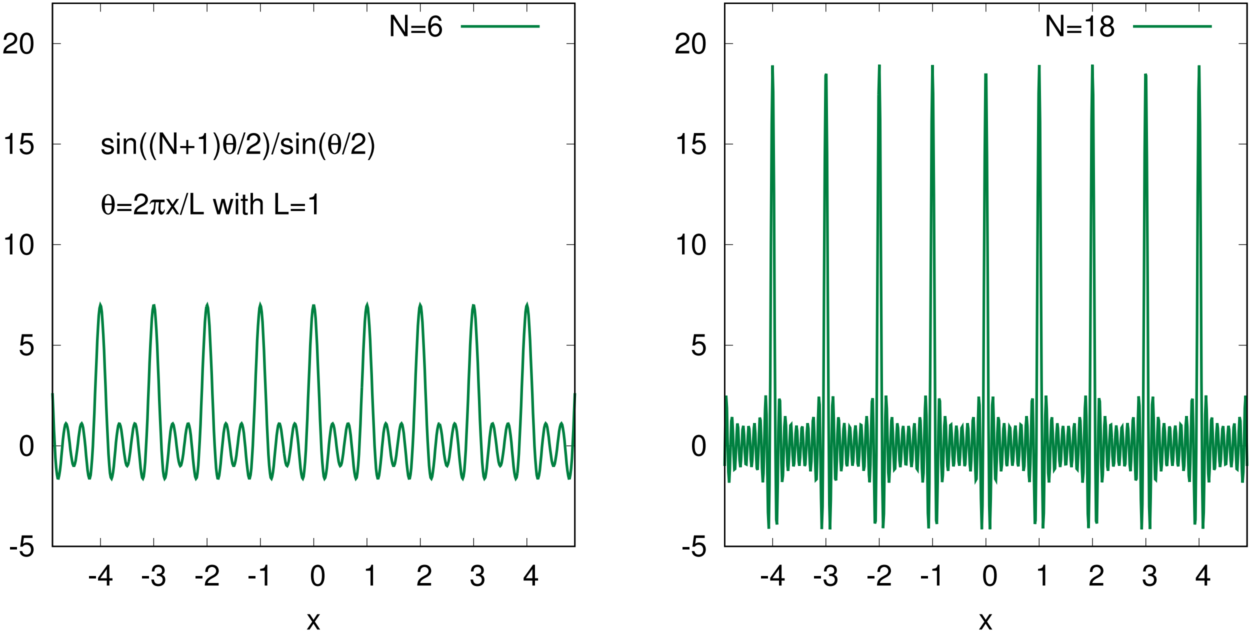
$$\lim_{N \to \infty} \delta_N(x) = \underbrace{\sum_{m=-\infty}^{\infty} \delta(x - mL)}_{\text{this is a } \delta\text{-fcn for } -L<x<L} \qquad m \text{ integer}$$

This proves completeness of the fourier series

We have shown more generally that

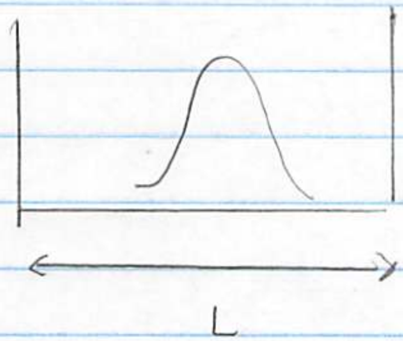
$$\frac{1}{L}\sum_n e^{i2\pi n x/L} = \sum_m \delta(x - mL)$$

This is the basis of the Poisson-Summation


N=6
sin((N+1)θ/2)/sin(θ/2)
θ=2πx/L with L=1
20
15
10
5
0
-5
-4
-3
-2
-1
0
1
2
3
4
x
N=18
20
15
10
5
0
-5
-4
-3
-2
-1
0
1
2
3
4
x

Fourier Transforms

Now



- Now take the box $\rightarrow \infty$, but keep k_n fixed

$$f(x) = \frac{1}{L} \sum_n f_n \, e^{i k_n x} \qquad k_n = \frac{2\pi n}{L}$$

- So k-becomes kind of continuous. $\Delta k = \frac{2\pi}{L}$ increm is very small

$$\frac{1}{L} \sum_n = \sum_n \frac{\Delta k}{2\pi} \Rightarrow \int_{-\infty}^{\infty} \frac{dk}{2\pi} \qquad f_n = f(k)$$

So

$$f(x) = \int \frac{dk}{2\pi} \, f(k) \, e^{ikx}$$

$$f(k) = \int_{-\infty}^{\infty} dx \, e^{-ikx} \, f(x)$$

The completeness says

$$\frac{1}{L}\sum_n e^{i\Delta k n x} = \sum_m \delta(x - mL)$$

which becomes $\frac{1}{L}\sum_n = \int \frac{dk}{2\pi}$ and $k = n\Delta k$. So we find

$$\int \frac{dk}{2\pi} e^{ik(x-y)} = \delta(x-y)$$

and $$\int dx\, e^{-i(k-k')x} = 2\pi\, \delta(k-k')$$

Summarizing $\langle x|k\rangle = e^{ikx}$ $\quad \langle k|x\rangle = e^{-ikx}$

k-space $\quad f(k) = \int dx\, e^{-ikx} f(x) = \langle k|f\rangle$

x-space $\quad f(x) = \int \frac{dk}{2\pi} e^{ikx} f(x)$

orthog $\quad \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = 2\pi\, \delta(k-k')$

complete $\quad \int \frac{dk}{2\pi} e^{+ik(y-x)} = \delta(y-x)$