A few of FT examples

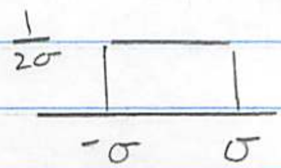
The constants are chosen so that $\int dx f(x) = 1 = \hat{f}(k)\Big|_{k=0}$

- $f(x) \longleftrightarrow \hat{f}(k)$

- $\dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \longleftrightarrow e^{-\frac{1}{2}\sigma^2 k^2}$

- $\dfrac{1}{2\sigma} e^{-|x|/\sigma} \longleftrightarrow \dfrac{1}{(k\sigma)^2 + 1}$

- Step fcn of integral $\underset{\int dx f(x) = 1}{1}$ $\longleftrightarrow \dfrac{\sin(k\sigma)}{k\sigma}$



Generally

$$\frac{1}{|\sigma|} f(x/\sigma) \longleftrightarrow \hat{f}(k\sigma)$$

behavior under scaling

Prf:

$$\text{FT of } \frac{1}{|\sigma|} f(x/\sigma) = \int_{-\infty}^{\infty} dx\, e^{-ikx} \frac{1}{|\sigma|} f(x/\sigma) = \int du e^{-ik\sigma u} f(u)$$

Change variables:

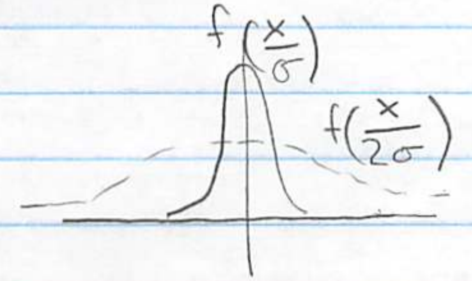
Don't loose absolute value

$$u = x/\sigma \qquad e^{ikx} = e^{ik\sigma u} \qquad du = \frac{dx}{|\sigma|}$$

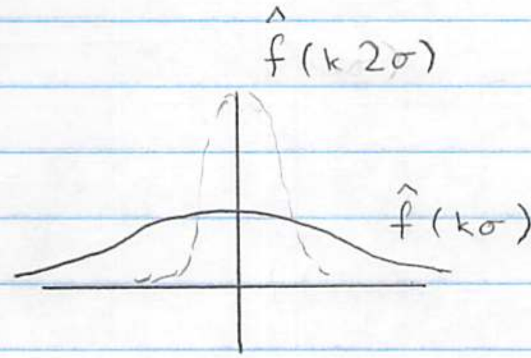
Then

$$\text{FT of } \frac{1}{|\sigma|} f(x/\sigma) = \hat{f}(k\sigma)$$

Picture:



coordinate space



k - space

i.e. narrower in x-space $\longleftrightarrow$ wider k-space

Fourier Transform of Convolutions

Consider two functions $f(x)$ and $g(x)$ and corresponding fourier transforms $\hat{f}(k)$ and $g(k)$. The convolution of f and g is

$$(f \circ g)(x) \equiv \int dx' \, f(x - x') \, g(x')$$

Its fourier transform is just the product

$$(\text{FT of } f \circ g)(k) = \hat{f}(k)\, \hat{g}(k)$$

of fourier transforms

Prf

- Use a shorthand notion

$$\int_x \equiv \int_{-\infty}^{\infty} dx \qquad \int_k = \int_{-\infty}^{\infty} \frac{dk}{2\pi}$$

$$\delta_{xy} \equiv \delta(x-y) \qquad \delta_{k_1 k_2} = \delta(k_1 - k_2)$$

Thus for example:

$$\int_k \int_x e^{ikx} = \int_k 2\pi\,\delta(k) = 1$$

i.e.

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx\; e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, 2\pi\,\delta(k) = 1$$

Then Proof of Theorem

$$FT(k) \equiv \int_x e^{-ikx} \int_{x_1} f(x-x_1)\, g(x_1)$$

Use

$$\int_{k_1} e^{ik_1(x-x_1)}\, \hat{f}(k_1) = f(x-x_1)$$

$$\int_{k_2} e^{ik_2 x_1}\, \hat{g}(k_2) = g(x_1)$$

So

$$FT(k) = \int_{k_1 k_2} \int_{x x_1} \hat{f}(k_1)\,\hat{g}(k_2)\, e^{-ikx}\, \underbrace{e^{ik_1(x-x_1)}}\, \underbrace{e^{ik_2 x_1}}$$

x is integrated over and this gives $2\pi\delta(k-k_1)$

Integrating over x and x_1 gives

$$FT(k) = \int_{k_1 k_2} f(k_1)\, g(k_2)\; 2\pi\,\delta_{k k_1}\; 2\pi\,\delta_{k_1 k_2}$$

e.g. $\int \frac{dk_1}{2\pi} 2\pi\delta(k-k_1)$ simply replaces k_1 with k

$$FT(k) = f(k)\, g(k)$$

Parseval's Theorem Again

From the convolution theorem, it may be shown

- $$\int \frac{dk}{2\pi}\, \hat{f}^*(k)\,\hat{g}(k) = \int dx\, f^*(x)\, g(x) = \langle f, g \rangle$$

This may be shown using the properties under complex conjugation of the Fourier transform, see handout.

- This says that the inner-product may be taken in coordinate or k-space.

Definition. Here $f(x)$ is a function and $\hat{f}(k) = \int_x e^{-ikx} f(x)$ is its Fourier transform

$$f(x) \Leftrightarrow \hat{f}(k) \tag{1}$$

Complex and Symmetry Properties. Here even and odd functions satisfy $f(-x) = \pm f(x)$:

$$f(-x) \Leftrightarrow \hat{f}(-k) \tag{2}$$

$$(f(x))^* \Leftrightarrow (\hat{f}(-k))^* \tag{3}$$

$$\text{if } f(x) \text{ is real} \Leftrightarrow \hat{f}(-k) = (\hat{f}(k))^* \tag{4}$$

$$\text{if } f(x) \text{ is even} \Leftrightarrow \hat{f}(k) \text{ is even} \tag{5}$$

$$\text{if } f(x) \text{ is odd} \Leftrightarrow \hat{f}(k) \text{ is odd} \tag{6}$$

$$\text{if } f(x) \text{ is real and even} \Leftrightarrow \hat{f}(k) \text{ is real and even} \tag{7}$$

$$\text{if } f(x) \text{ is real and odd} \Leftrightarrow \hat{f}(k) \text{ is imaginary and odd} \tag{8}$$

Shift properties:

$$e^{ik_o x} f(x) \Leftrightarrow \hat{f}(k - k_o) \tag{9}$$

$$f(x - x_o) \Leftrightarrow e^{-ikx_o} \hat{f}(k) \tag{10}$$

Proof of Eq (3)

$$f(x) = \int_k e^{ikx} \hat{f}(k)$$

$$f^*(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \hat{f}^*(k)$$

Change variables $\bar{k} = -k$

$$f^*(x) = \int_{-\infty}^{\infty} \frac{d\bar{k}}{2\pi} e^{i\bar{k}x} \hat{f}^*(-\bar{k})$$

Thus

$$f^*(x) \Longleftrightarrow \hat{f}^*(-k)$$

Proof of Eq (9)

$$\text{FT of } e^{ik_0x} f(x) = \int dx\, e^{-ikx} e^{ik_0x} f(x)$$

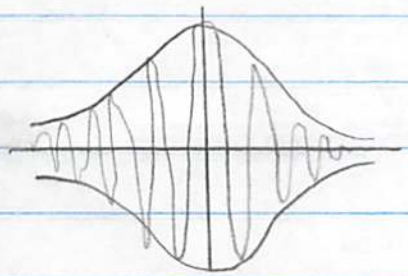
$$= \int dx\, e^{-i(k-k_0)x} f(x)$$

$$= \hat{f}(k-k_0)$$

Example 1

I am defining a "Gaussian function" G(x)=exp(-x^2)/Sqrt[2 Pi] here

A wave packet:



$$f(x) = \overbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}}^{\equiv \frac{1}{\sigma} G(x/\sigma)} \cos(k_0 x)$$

$$\lambda_0 = 2\pi/k_0 \ll \sigma \quad \text{and} \quad k_0 \gg \frac{1}{\sigma}$$

- Realize that since $f(x)$ is real and even $\hat{f}(k)$ is real and even

- Write $\cos(k_0 x) = \dfrac{e^{ik_0 x} + e^{-ik_0 x}}{2}$

The exp(ik0 x) factors will shift the function.

- Recognize that the FT of a gaussian is another Gaussian. But since the wave packet is broad in coordinate space it is narrow in k-space

this is supposed to read $G(x/\sigma)$

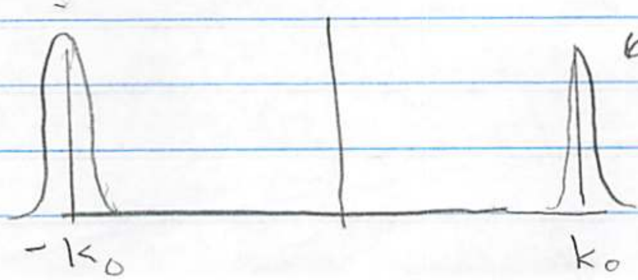
$$(\text{FT of } f(x))(k) = \text{FT of } \frac{1}{2\sigma} G\left(\frac{x}{\sigma}\right) e^{ik_0 x} + \frac{1}{2\sigma} G(x/\sigma) e^{-ik_0 x}$$

More explicitly the first term is $\frac{1}{2}e^{-\frac{1}{2}(k-k_0)^2\sigma^2}$

$$= \frac{1}{2} G((k-k_0)\sigma) + \frac{1}{2} G((k+k_0)\sigma)$$

Picture

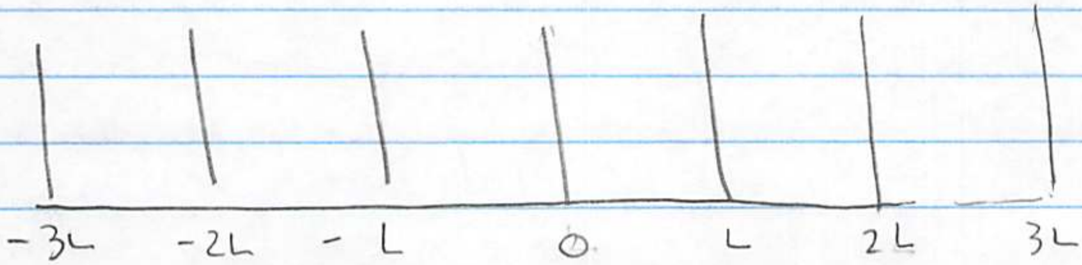
Gaussian of width $1/\sigma$



Example 2

Dirac Comb:

$$DC(x) = \sum_{m=-\infty}^{\infty} \delta(x - mL)$$



Recall that

$$DC(x) = \frac{1}{L} \sum_{n} e^{i \frac{2\pi n}{L} x}$$

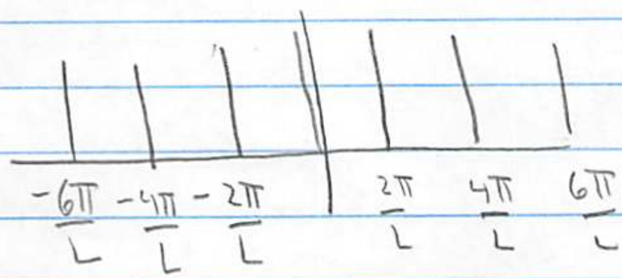
Then

$$DC(k) = \int_x e^{-ikx} \, DC(x)$$

$$= \frac{1}{L} \sum_{n} 2\pi \, \delta\left(k - \frac{2\pi n}{L}\right)$$

Note this is another k-space Dirac Comb:

Remember the fourier transform of a Dirac comb of spacing L is another dirac comb of spacing 2 Pi/L



Recall that

$$\int \frac{dk}{2\pi} \hat{f}^*(k)\, \hat{g}(k) = \int dx\, f^*(x)\, g(x)$$

Apply this with $f(x) = DC(x)$ $\hat{f}^*(k) = DC(k)$

- $$\int \frac{dk}{2\pi} DC(k)\, \hat{g}(k) = \int dx\, DC(x)\, g(x)$$

$$\frac{1}{L} \sum_n \hat{g}\left(\frac{2\pi n}{L}\right) = \sum_m g(mL)$$

- This is known as the Poisson summation formula

- If a sum is slowly convergent in coordinate space, it may be easier to perform the sum in k-space. This is known as Ewald summation