1.5 Constraints

Lagrange multipliers

- First we considered minimizing $U(x,y)$ subject to a constraint $Q(x,y)=0$. We said that we should instead minimize
$$\hat{U}(x,y,\lambda) = U(x,y) - \lambda Q(x,y)\,. \tag{1.104}$$
λ is known as a Lagrange multiplier[5]. This leads to the conditions
$$d\hat{U}(x,y) = \left(\frac{\partial U}{\partial x} - \lambda\frac{\partial Q}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y} - \lambda\frac{\partial Q}{\partial y}\right)dy - Qd\lambda = 0 \tag{1.105}$$
where the terms in front of dx, dy, and $d\lambda$ should be set to zero. We explained that Q can be thought of as a generalized coordinate, and λ is a generalized force conjugate to Q. This is just like adding an external force. For instance if I have a potential $U(x,y)$ and add an external force f in the x direction then the new potential is
$$\hat{U}(x,y,f) = U(x,y) - fx\,. \tag{1.106}$$
The forces of constraint in the x and y directions are
$$F_x = \lambda\partial_x Q\,, \tag{1.107}$$
$$F_y = \lambda\partial_y Q\,. \tag{1.108}$$

- The setup easily generalizes to more coordinates and more constraints. For coordinates x^A and constraints $Q^\alpha(x^A)$ with $\alpha = 1\ldots m$, if we want to minimize $U(x^A)$ subject to these constraints, we instead extremize
$$\hat{U}(x^A) = U(x^A) - \lambda_\alpha Q^\alpha(x^A) \tag{1.109}$$
requiring that $d\hat{U}=0$, i.e. require
$$\frac{\partial \hat{U}}{\partial x^A} = 0 \tag{1.110}$$
$$\frac{\partial \hat{U}}{\partial \lambda_\alpha} = 0 \tag{1.111}$$
The forces of constraint in the x^A direction are
$$F_A = \lambda_\alpha \frac{\partial Q^\alpha}{\partial x^A} \tag{1.112}$$

Newton's Laws and Lagrange with constraints

- Consider Newton's Laws for particles with positions $\boldsymbol{r}_a$. For simplicity consider just one constraint.
$$Q(\boldsymbol{r}_a) = 0 \tag{1.113}$$
Then
$$dQ = \nabla_{\boldsymbol{r}_a} Q \cdot d\boldsymbol{r}_a = 0 \tag{1.114}$$
The forces of constraints$\boldsymbol{F}_a^C$ do no work
$$\boldsymbol{F}_a^C \cdot d\boldsymbol{r}^a = 0 \tag{1.115}$$

[5] The sign in front of λ is irrelevant. The choice here is so that λ corresponds to the generalized force in the direction of increasing Q, compare to Eq. (1.106). When we consider contraints in the Lagrangian, $L = T - U$, the multipliers will then come with a plus sign $\hat{L} = T - U + \lambda Q$.

Thus, we make take $\boldsymbol{F}_a^C$ to be proportional to the gradient of Q

$$\boldsymbol{F}_a^C = \lambda \nabla_{\boldsymbol{r}_a} Q \tag{1.116}$$

Then Newton's Laws read

$$\frac{d\boldsymbol{p}_a}{dt} = \boldsymbol{F}_a^{\text{ext}} + \lambda \nabla_{\boldsymbol{r}_a} Q\,. \tag{1.117}$$

Then Newton's Law ($\boldsymbol{F} = m\boldsymbol{a}$) *and* the constraint, determine the accelerations of the particles *and* the magnitude of the forces of constraint, i.e. λ.

- You should do some simple problems on Attwood's machines (see below) to convice yourself that we are always solving Eq. (1.117) when doing Freshmann physics problems.

- In the Lagrangian formalism we add some lagrange multipliers to enforce the constraints. Instead of extremizing $L(\dot{\boldsymbol{r}}_a, \boldsymbol{r}_a)$, one extremizes $\hat{L}(\dot{\boldsymbol{r}}_a, \boldsymbol{r}_a, \lambda) = L + \lambda Q$, where λ is like an extra coordinate. The Euler-Lagrange equations for $\hat{L}$ are[6]

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{\boldsymbol{r}}_a}\right) = \frac{\partial \hat{L}}{\partial \boldsymbol{r}_a} \tag{1.120}$$

$$0 = Q \tag{1.121}$$

- If there are more constraints Q^α, simply make the replacement $\lambda Q \to \lambda_\alpha Q^\alpha$ in the lagrangian formalism. In the Newtonian formalism the force of constraint on the a-th particle is

$$\boldsymbol{F}_a = \lambda_\alpha \nabla_{\boldsymbol{r}_a} Q^\alpha\,. \tag{1.122}$$

- *Attwood machine.* Consider two masses m_1 and m_2 hanging over a massless pulley (you know the problem!). We have two coordinates z_1 and z_2 where z_1 and z_2 are the distances below the pulley (increasing z means further down). The constraint is

$$Q = z_1 + z_2 - L \tag{1.123}$$

The hatted Lagrangian is

$$\hat{L} = \tfrac{1}{2} m_1 \dot{z}_1^2 + \tfrac{1}{2} m_2 \dot{z}_2^2 + m_1 g z_1 + m_2 g g z_2 + \lambda(z_1 + z_2 - L) \tag{1.124}$$

Newton's or Lagranges' equation of motion are

$$m_1 a_1 = m_1 g + \lambda \tag{1.125}$$

$$m_2 a_2 = m_2 g + \lambda \tag{1.126}$$

$$z_1 + z_2 = L \tag{1.127}$$

Which are easily solved for a_1, a_2 and λ, using that Eq. (1.127) implies by differentationg that $a_1 + a_2 = 0$. Solving these equations gives λ negative, i.e. the force is up not down. The case when the pulley has mass in the Lagrangian formalism is suggested as an excercise.

[6] Perhaps we should write it a bit more explicitly. The coordinates of $\boldsymbol{r}_a$ are r_a^i with $i = x, y, z$. We mean

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{r}_a^i}\right) = \frac{\partial \hat{L}}{\partial r_a^i} \tag{1.118}$$

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{\lambda}}\right) = \frac{\partial \hat{L}}{\partial \lambda} \tag{1.119}$$

The equation $0 = Q$ follows from the equation for λ, which simply enforces the constraint.