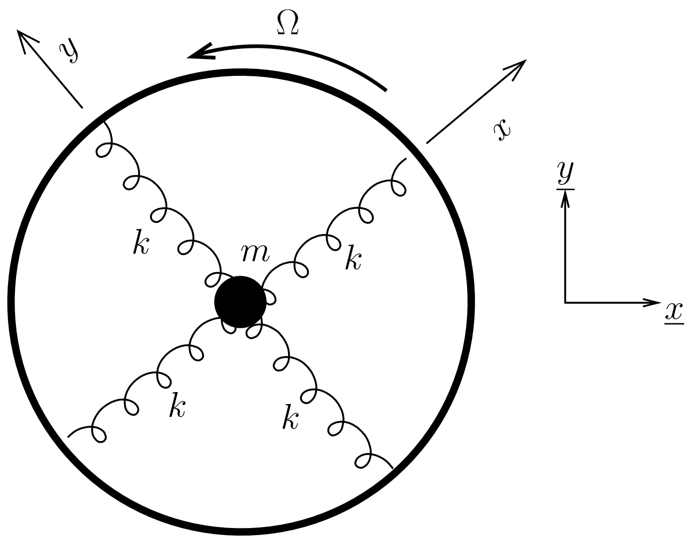
Midterm 2019

1 A rotating contraption

Consider a particle of mass m moving in the horizontal plane. The particle is connected to a circular frame of radius a via four springs which are unstretched when the particle is in the center of the frame. The springs have spring constant k. The frame rotates as a whole with constant angular velocity Ω. At all times the particle is only slightly displaced from the center of the frame, and thus a harmonic approximation is always valid.



(a) (2 points) Briefly explain the following formula for the time derivative of the particles position vector $\boldsymbol{r}(t)$ in the plane relative to the origin (the center):

$$\frac{d\boldsymbol{r}}{dt} = \left(\frac{d\boldsymbol{r}}{dt}\right)_r + \boldsymbol{\Omega} \times \boldsymbol{r}\,. \tag{1}$$

Here $d\boldsymbol{r}/dt$ is the velocity of the particle in the lab frame (the $\underline{x}$, $\underline{y}$ coordinate system), $(d\boldsymbol{r}/dt)_r$ is the velocity of the particle in the rotating frame (the x, y coordinate system), and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ points out of the page.

(b) (8 points) Write down the Lagrangian of the system using (co-rotating) coordinates x, y. Also write the Lagrangian in co-rotating cylindrical coordinates $r = \sqrt{x^2+y^2}, \phi = \tan^{-1} y/x$. The cylindrical version will be most convenient in what follows.

(c) (8 points) Construct the Hamiltonian of the system (using the r, ϕ coordinates and associated momentum) and identify any constants of the motion.

(d) (2 points) Determine Hamilton's equation of motion for r and ϕ and associated momenta. Show in particular that the equation of motion for r takes the form

$$\frac{dr}{dt} = \frac{p_r}{m} \tag{2}$$

$$\frac{dp_r}{dt} = -\frac{\partial V_{\text{eff}}}{\partial r} \tag{3}$$

for an appropriate effective potential.

(e) (4 points) Find a configuration where the radius is r is constant in time, and determine how the angle $\phi(t)$ evolves in the rotating and lab frames for this configuration.

(f) (2 points) Give a physical interpretation for the constant radius r and $\phi(t)$ of part (e) by drawing a free body diagram in the rotating frame.

Solution:

(a) We have the vector $\boldsymbol{r} = r^a \boldsymbol{e}_a(t)$, where $\boldsymbol{e}_a(t)$ is a rotating basis. Then the derivative is

$$\frac{d\boldsymbol{r}}{dt} = \frac{dr^a}{dt}\boldsymbol{e}_a + r^a \frac{d\boldsymbol{e}_a}{dt} \,. \tag{4}$$

The first term is the change of $\boldsymbol{r}$ as viewed by a an observer rotating with the frame, $(d\boldsymbol{r}/dt)_r$. The second term is the change in the rotating basis with time. A little geometry shows $d\boldsymbol{e}_a/dt = \Omega \times \boldsymbol{e}_a$. Thus, since $r^a\boldsymbol{e}_a = \boldsymbol{r}$ we have

$$\frac{d\boldsymbol{r}}{dt} = \left(\frac{d\boldsymbol{r}}{dt}\right)_r + \boldsymbol{\Omega} \times \boldsymbol{r} \,. \tag{5}$$

More explicitly the velocity is

$$\boldsymbol{v} = \dot{x}\boldsymbol{e}_1 + \dot{y}_2\boldsymbol{e}_2 - \Omega y \boldsymbol{e}_1 + \Omega x \boldsymbol{e}_2 \tag{6}$$

(b) The potential energy in terms of the co-rotating coordinates, x, y, is simply

$$U = \frac{1}{2}(2k)(x^2 + y^2) = \frac{1}{2}m\omega^2(x^2 + y^2) \tag{7}$$

where

$$\omega_0^2 = \frac{2k}{m} \tag{8}$$

You can derive this just by pretending that the frame is not rotating.

Then the Lagrangian is simply

$$L = \frac{1}{2}mv^2 - U(x, y) \tag{9}$$

Substituting the results of part (a), the kinetic term reads

$$\frac{m}{2}v^2 = \frac{m}{2}(\dot{x} - \Omega y)^2 + \frac{m}{2}(\dot{y} + \Omega x)^2 \,. \tag{10}$$

Then in cartesian coordinates we have

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - m\Omega(\dot{x}y - \dot{y}x) + \frac{1}{2}m\Omega^2(x^2 + y^2) - U(x) \tag{11}$$

In terms of cylindrical coordinates

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \Omega m r^2 \dot{\phi} + \frac{1}{2}m(\Omega^2 - \omega_0^2)r^2 \tag{12}$$

(c) Then the Hamiltonian is

$$H = p_r\dot{r} + p_\phi\dot{\phi} - L \,. \tag{13}$$

With $p_r = m\dot{r}$ and

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2(\dot{\phi} + \Omega) \qquad \dot{\phi} = \frac{p_\phi - mr^2\Omega}{mr^2} \tag{14}$$

we find

$$H = p_r\left(\frac{p_r}{m}\right) + p_\phi\left(\frac{p_\phi - mr^2\Omega}{mr^2}\right) - \left[\left(\frac{p_r^2}{2m} + \frac{1}{2}\frac{(p_\phi - mr^2\Omega)^2}{mr^2}\right)\right] \\ - mr^2\Omega\frac{(p_\phi - mr^2\Omega)}{mr^2} + \frac{1}{2}m(\Omega^2 - \omega_0^2)r^2\,. \tag{15}$$

Leading to

$$H = \frac{p_r^2}{2m} + \frac{(p_\phi - mr^2\Omega)^2}{2mr^2} + \frac{1}{2}m(\omega_0^2 - \Omega^2)r^2 \tag{16a}$$

$$= \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + \frac{1}{2}m\omega r^2 - p_\phi\Omega \tag{16b}$$

We make the following comments:

- In general the Hamiltonian in a rotating frame is

$$H_{\rm rot} = H_{\rm non-rot} - \boldsymbol{\Omega}\cdot\boldsymbol{L} \tag{17}$$

  where $\boldsymbol{L}$ is the angular momentum in the non-rotating frame. This is clearly reflected in Eq. (16)

- This problem is a good example of a general class of lagrangians:

$$L = \sum_i \frac{1}{2}a_i\,(\dot{q}^i)^2 + b_i\,\dot{q}^i - U(q) \tag{18}$$

  where $a_i(q)$ and $b_i(q)$ are functions of the coordinates. The Hamiltonian for this system is

$$H = \sum_i \frac{(p_i - b_i)^2}{2a_i} + U(q) \tag{19}$$

  and the first integral (hamiltonian function) is independent of $b_i(q)$

$$h = \frac{1}{2}a_i(\dot{q}^i)^2 + U(q) \tag{20}$$

The first evident integral of motion the momentum conjugate to ϕ, since the coordinate ϕ is cyclic:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2(\dot{\phi} + \Omega) \equiv \ell \tag{21}$$

Also the Lagrangian is not an explicit function of time, and thus the system has a first integral:

$$E = h = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{1}{2}m(\omega^2 - \Omega^2)r^2 \tag{22a}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{(\ell - mr^2\Omega)^2}{2mr^2} + \frac{1}{2}m(\omega^2 - \Omega^2)r^2 \tag{22b}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 - \ell\Omega \tag{22c}$$

Again we see the general expected result

$$E_{\rm rot} = E_{\rm non-rot} - \ell\Omega \tag{23}$$

(d) The equation of motion of ϕ reads

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \tag{24a}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi - mr^2\Omega}{mr^2} \tag{24b}$$

$$= \frac{\ell}{mr^2} - \Omega \tag{24c}$$

The equation of motion of r reads

$$\frac{dr}{dt} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \tag{25a}$$

$$\frac{dp_r}{dt} = -\frac{\partial H}{\partial r} = -\partial_r V_{\rm eff}(r) \tag{25b}$$

with

$$V_{\rm eff} = \frac{\ell^2}{2mr^2} + \frac{1}{2}mr^2\omega^2 + \text{const} \tag{26}$$

(e) The minimum of the effective potential determines an orbit of constant radius $r = r_0$

$$0 = \frac{\partial V}{\partial r} \tag{27a}$$

$$0 = -\frac{\ell^2}{mr^3} + m\omega^2 r \tag{27b}$$

which yields

$$r_0^2 = \frac{|\ell|}{m\omega} \tag{28}$$

Then

$$\dot{\phi} = \frac{\ell}{mr_0^2} + \Omega \tag{29a}$$

$$= \frac{\ell}{|\ell|}\omega + \Omega \tag{29b}$$

(d) A quick free body diagram in the rotating frame gives insight. The forces are the coriolis force, the centripetal force, and the force from the spring. The magnitude of velocity in the rotating frame is $v_0 = \dot{\phi}_0 r_0$ with $\dot{\phi}_0$ a constant, i.e. the particle is moving in a circle at radius r_0 in the rotating frame.

The coriolis force is

$$\vec{F}_{cor} = -2m\boldsymbol{\Omega} \times \boldsymbol{v} = -2m\dot{\phi}_0 \, r_0 \, \Omega \, \hat{\boldsymbol{r}} \, . \tag{30}$$

The centripetal force is

$$\vec{F}_{centrip} = m\Omega^2 r_0 \, \hat{\boldsymbol{r}} \, . \tag{31}$$

The force from the springs is

$$\vec{F}_{spring} = -m\omega^2 r \, \hat{\boldsymbol{r}} \, . \tag{32}$$

Then the sum of forces in the rotating frame is equal to the mass times the acceleration in this frame. In the accelerating frame the particle is moving in a circle with velocity $v_0 = \dot{\phi}_0 r_0$ and therefore has acceleration in the rotating frame is

$$\boldsymbol{a}_r = -\frac{v_0^2}{r_0} = -\dot{\phi}_0^2 r_0 \, \hat{\boldsymbol{r}} \tag{33}$$

Then Newton's law in the rotating frame reads

$$\vec{F}_{cor} + \vec{F}_{centrip} + \vec{F}_{spring} = \left(-2m\dot{\phi}_0 r_0 \Omega + m\Omega^2 r_0 - m\omega^2 r_0 \right) \hat{\boldsymbol{r}} = -m\dot{\phi}_0^2 r_0 \, \hat{\boldsymbol{r}} = m\boldsymbol{a}_r \tag{34}$$

So this is satisfied if

$$\dot{\phi}_0^2 - 2\dot{\phi}_0 \Omega + \Omega^2 - \omega^2 = 0 \tag{35}$$

Solving the quadratic equation for ϕ_0 we have

$$\dot{\phi}_0 = \Omega \pm \sqrt{\Omega^2 - (\Omega^2 - \omega^2)} \, , \tag{36}$$

$$= \Omega \pm \omega \, . \tag{37}$$

The two roots of the quadratic equation are the allowed values of ϕ given in Eq. (29), and correspond to wether the particle moves in a clockwise or counter-clockwise fashion in the rotating frame. In the lab frame

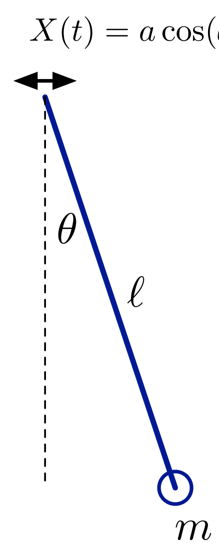
$$(\dot{\phi}_0)_{\mathrm{lab}} = \pm\omega \tag{38}$$

which says that the particle just goes in a circle as if the frame was not there.

Midterm 2019

2 A horizontally driven pendulum

A pendulum consists of a massless rod of length ℓ with a mass m at its end. The pendulum swings with an angle $\theta(t)$ relative to the vertical in the earth's gravitational field. The support (or pivot point) of the pendulum is driven horizontally with x coordinate $X(t) = a\cos\omega t$ as shown below, and $a/\ell \ll 1$.



(a) (8 points) Write down the Lagrangian of the system and determine the equations of motion. Do not assume that θ is small.

Hint: Make sure your Lagrangian and equations of motion are correct. All subsequent steps depend on these being right.

(b) (4 points) Interpret the equations of motion of (a) by writing down Newton's Laws in an accelerating frame which moves horizontally from side to side with the pivot point, i.e. the origin of the accelerating frame has position $X(t)$ in the lab frame. Draw a free body diagram in this accelerating frame to support your interpretation.

(c) (6 points) For a small frequency $\omega \ll \omega_0$ (with $\omega_0^2 \equiv g/\ell$), the system will make small oscillations around $\theta = 0$. Find the first non-linear correction to the oscillations. Sketch the power spectrum, i.e. if $\theta(t) = \sum_n c_n e^{in\omega t}$ qualitatively sketch $|c_n|^2$ versus n.

(d) (4 points) When the system comes close to resonance, non-linear effects will become important. Estimate the frequency shift from resonance $\Delta\omega \equiv (\omega - \omega_0)$ where non-linear corrections become of order unity.

Solution: (a) We write the position of the pendulum

$$x = a\cos\omega t + \ell\sin\theta \tag{39}$$
$$y = -\ell\cos\theta \tag{40}$$

Then

$$\dot{x} = -a\omega\sin\omega t + \ell\cos\theta\dot{\theta} \tag{41}$$
$$\dot{y} = \ell\dot{\theta}\,\sin\theta \tag{42}$$

So the Lagrangian

$$L = \frac{1}{2}mv^2 - mgy \tag{43}$$

reads

$$L = \frac{m}{2}(-a\omega\sin(\omega t) + \ell\dot{\theta}\cos\theta)^2 + \frac{m}{2}(\ell\dot{\theta}\sin\theta)^2 + mg\ell\cos\theta \tag{44}$$

Leading to

$$L = \frac{m}{2}\ell^2\dot{\theta}^2 - ma\omega\sin(\omega t)\ell\cos\theta\dot{\theta} + mg\ell\cos\theta \tag{45}$$

Integrating by parts, $d/dt\sin(\theta) = \cos\theta\dot{\theta}$ we have

$$L = \frac{m}{2}\ell^2\dot{\theta}^2 + ma\omega^2\cos(\omega t)\ell\sin(\theta) + mg\ell\cos\theta \tag{46}$$

The equation of motion reads

$$\frac{d}{dt}(m\ell^2\dot{\theta}) = -mg\ell\sin\theta + m(a\omega^2\cos\omega(t))\ell\cos\theta \tag{47}$$

(b) In the accelerating frame there is an effective force in the x-direction

$$F_X = -m\ddot{X} = ma\omega^2\cos(\omega t) \tag{48}$$

This gives rise to a torque

$$\tau_X = F_X(t)\ell\cos\theta \tag{49}$$

The change in angular momentum $m\ell^2\dot{\theta}$ is due to the gravitational torque $\tau_g = -mg\ell\sin\theta$ and the effective torque τ_X.

(c) One is looking for a steady state solution. For small oscillations $\cos\theta \simeq 1 - \theta^2/2$, the equation of motion including the first non-linear correction reads

$$\ddot{\theta} = -\omega_0^2\theta + f_0\cos\omega t - f_0\cos(\omega t)\frac{\theta^2}{2} \tag{50}$$

with $f_0 \equiv \omega^2 a/\ell$. The rotating wave approximation writes

$$\theta^{(0)}(t) = A\cos(\omega t + \varphi) + \theta^{(1)}(t) \tag{51}$$

and solves for A and φ. Here we are looking for the steady state solution, so A and φ are constants. We will try to find a solution with $\varphi = 0$.

First examine the non-linear term, which reads

$$- f_0 \cos(\omega t) \frac{\theta^2}{2} = -\frac{1}{2} f_0 A^2 \cos(\omega t)^3 \tag{52}$$

The $\cos(\omega t)^3$ term is written:

$$\cos(\omega t)^3 = \frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos(\omega t) \tag{53}$$

In a rotating wave approximation we keep only the $\cos(\omega t)$ term and treat later $\cos(3\omega t)$ as a perturbation.

The equation of motion without approximation reads

$$\begin{aligned}\left[-\omega^2 A\cos(\omega t) + \omega_0^2 A\cos(\omega t) + f_0\cos(\omega t) + \frac{3f_0}{8}A^2\cos(\omega t)\right]& \\ + \left[\ddot{\theta}^{(1)} + \omega_0^2\theta^{(1)} + \frac{f_0A^2}{8}\cos(3\omega t)\right] &= 0\end{aligned} \tag{54}$$

Comparing the $\cos(\omega t)$ terms we have

$$(-\omega^2 + \omega_0^2)A + \frac{3}{8}f_0A^2 = f_0 \tag{55}$$

This is solved by iteration

$$A = A^{(0)} + A^{(1)}\,, \tag{56}$$

leading to

$$A^{(0)} = \frac{f_0}{-\omega^2 + \omega_0^2}\,, \tag{57}$$

and then

$$A^{(1)} = \frac{3}{8}\frac{f_0^3}{(-\omega^2 + \omega_0^2)^3}\,. \tag{58}$$

Then from the equation for $\theta^{(1)}$

$$\left[\ddot{\theta}^{(1)} + \omega_0^2\theta^{(1)} + \frac{f_0A^2}{8}\cos(3\omega t)\right] = 0 \tag{59}$$

we substitute $\theta^{(1)}(t) = B\cos(3\omega t)$ and find

$$\theta^{(1)} = \frac{1}{8}\frac{f_0^3}{(-9\omega^2 + \omega_0^2)(-\omega^2 + \omega_0^2)^2}\cos(3\omega t) \tag{60}$$

(d) The expansion parameter in the above expressions is

$$\frac{f_0}{-\omega^2 + \omega_0^2}\,. \tag{61}$$

Clearly the non-linear terms become of order unity when $A^{(1)} \sim A^{(0)}$. This happens when the expansion parameter is no longer small

$$\frac{f_0}{-\omega^2 + \omega_0^2} \sim 1 \,. \tag{62}$$

Writing $\omega = \omega_0 - \Delta\omega$ and solving for $\Delta\omega$ we find

$$\frac{f_0}{2\omega_0 \Delta\omega} \sim 1 \tag{63}$$

i.e. when the frequency shift away from resonance is of order[1]

$$\Delta\omega \sim \frac{f_0}{2\omega_0} \sim \omega_0 \frac{a}{\ell} \tag{64}$$

then the non-linear effects can no longer be treated perturbatively. At this point one needs to examine how the non-linearities shift the position resonance, and give rise to the hysteresis phenomenon discussed in class.

[1]where we have recalled that $f_0 = \omega^2 a/\ell$. In the last step we recognized that we are close to resonance, and thus $\omega \simeq \omega_0$.

Final 2019

1 Double pendulums

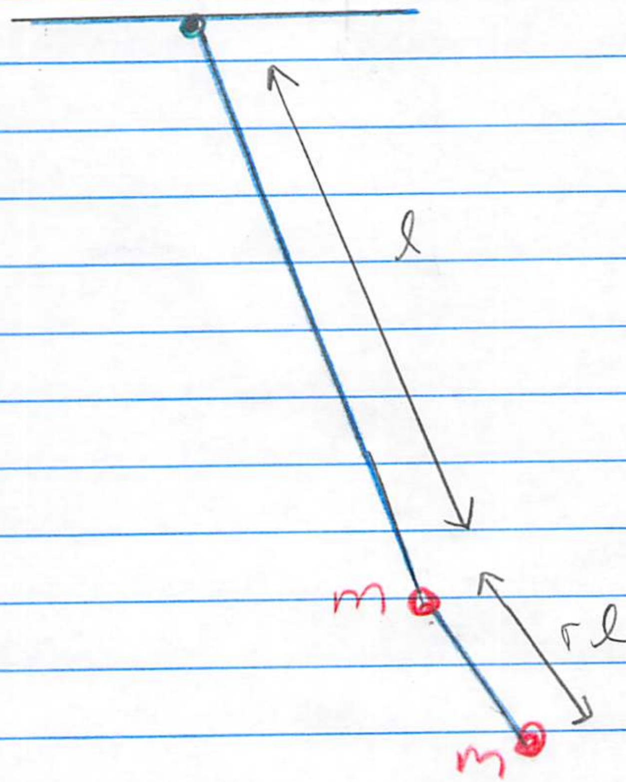
Consider two light rods coupled together with a hinge. The length of the first rod is ℓ, and it has a mass m at its end (see figure). The length of the second rod is a factor r of smaller than the first (i.e. its length is $r\ell$), and it also has mass m on its end. The system oscillates harmonically in the earth's gravitational field (see figure).

(a) Write down the Lagrangian of the system in a harmonic approximation. Check your work.

(b) Determine frequencies of the normal modes.

(c) Describe physically what is going on in the limit when $r \ll 1$.

Double Pendulums



Solution

(a) Writing the coordinates with small angle approximations $\sin\theta = \theta$, and $\cos\theta = 1-\theta^2/2$ we have

$$x_1 = \ell\theta_1 \tag{1}$$

$$y_1 = -\ell + \ell\frac{\theta_1^2}{2} \tag{2}$$

$$x_2 = \ell\theta_1 + r\ell\theta_2 \tag{3}$$

$$y_2 = -\ell + \ell\frac{\theta_1^2}{2} + -r\ell + r\ell\frac{\theta_1^2}{2} \tag{4}$$

We find

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{y}_2^2 - mgy_1 - mgy_2 \tag{5}$$

So up to constants and keeping only harmonic terms

$$L = \frac{1}{2}m\ell^2\dot{\theta}_1^2 + \frac{1}{2}m\ell^2(\dot{\theta}_1 + r\dot{\theta}_2)^2 - \frac{1}{2}mg\ell\theta_1^2 - \frac{1}{2}mg\ell(\theta_1^2 + r\theta_2^2) \tag{6}$$

Regrouping terms

$$L = \frac{1}{2}m\ell^2\left(2\dot{\theta}_1^2 + 2r\dot{\theta}_1\dot{\theta}_2 + r^2\dot{\theta}_2^2\right) - \frac{1}{2}mg\ell\left(2\theta_1^2 + r\theta_2^2\right) \tag{7}$$

(b) So the equation of motion takes the form

$$\begin{pmatrix} 2 & r \\ r & r^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = -\omega_0^2\begin{pmatrix} 2 & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \tag{8}$$

So we look for charactistic solution

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \vec{E}e^{-i\omega t} \tag{9}$$

Leading to the matrix equation

$$\begin{pmatrix} 2(-\omega^2 + \omega_0^2) & -r\omega^2 \\ -r\omega^2 & (-r^2\omega^2 + r\omega_0^2) \end{pmatrix}\vec{E} = 0 \tag{10}$$

So the non-trivial solutions are when the determinant to this matrix is zero. Evaluating this determinant leads to

$$2(-\omega^2 + \omega_0)^2(-\omega^2 + \omega_0^2/r) - \omega^4 = 0 \tag{11}$$

So

$$\omega^4 - 2(\omega_0^2 + \omega_0^2/r)\omega^2 + 2\omega_0^2(\omega_0^2/r) = 0 \tag{12}$$

The solutions are

$$\omega^2 = \left(\omega_0^2 + \omega_0^2/r \pm \sqrt{(\omega_0^2 + \omega_0^2/r)^2 - 2\omega_0^2(\omega_0^2/r)}\right) \tag{13}$$

So

$$\omega^2 = \omega_0^2\left(1 + u \pm \sqrt{1 + u^2}\right) \tag{14}$$

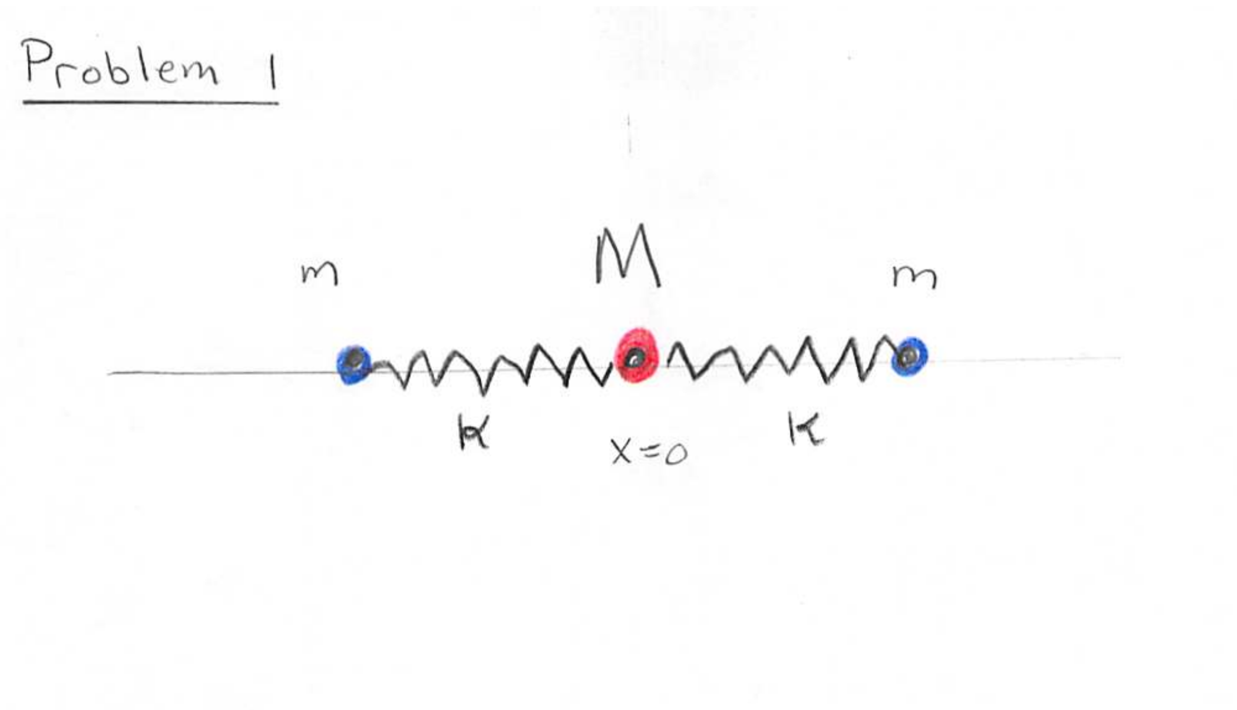
where $u = 1/r$

(c) In the limit that $r \to 0$ the center of mass swings comparatively slowly at expected freuency $\omega^2 = \omega_0^2$ due to the torque of the gravitational field. The internal oscillations are faster, and oscillates at frequency $\omega^2 = 2g/\ell_2$. One can simply take the limit that $u \to \infty$ yielding the two frequencies

$$\omega^2 \simeq \begin{cases} \omega_0^2 + \mathcal{O}(\omega_0^2/u) \\ \omega_0^2(2u+1) + \mathcal{O}(\omega_0^2/u) \end{cases} \tag{15}$$

1 A kicked oscillator

Consider three particles of masses m, M, m, constrained to move along the x-axis. The particles are connected by ideal unstretched springs with spring constant κ as shown below (see the last page).



Neglect friction. The unstretched distance between the particles is a.

(a) Determine the normal modes of the system, and make a sketch of them. Identify any zero modes.

(b) At time $t = 0$, the central particle (with mass M) is given an impulsive kick $F_0 = P_0\delta(t)$ to the right. Determine the the position of each particle as a function of time for $t > 0$.

Assume that the impulse is small enough that subsequent oscillations can be treated in a harmonic approximation.

Solution:

(a) We first write down the Lagrangian

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}M\dot{X}^2 - \frac{1}{2}\kappa(x_1 - x_2)^2 - \frac{1}{2}\kappa(x_2 - X)^2 \tag{1}$$

The equation of motion is

$$m\ddot{x}_1 = -\kappa(x_1 - x_2) \tag{2}$$
$$M\ddot{X} = \kappa(x_1 - X) - \kappa(x_2 - X) \tag{3}$$
$$m\ddot{x}_2 = +\kappa(x_2 - X) \tag{4}$$

which in matrix form reads

$$\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{X} \\ \ddot{x}_2 \end{pmatrix} = \kappa \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} \tag{5}$$

The most direct way is to find the normal modes is substitute a form

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = e^{-i\omega t} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{6}$$

leading to

$$\begin{pmatrix} \kappa - m\omega^2 & -\kappa & 0 \\ -\kappa & 2\kappa - M\omega^2 & -\kappa \\ 0 & -\kappa & \kappa - m\omega^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0\,. \tag{7}$$

Then these linear equations will have non-trivial solutions when the determinant vanishes

$$\det \begin{pmatrix} \kappa - m\omega^2 & -\kappa & 0 \\ -\kappa & 2\kappa - M\omega^2 & -\kappa \\ 0 & -\kappa & \kappa - m\omega^2 \end{pmatrix} = 0 \tag{8}$$

Leading to

$$-\omega^2(\kappa - m\omega^2)(\kappa(2m + M) - mM\omega^2) = 0 \tag{9}$$

So

$$\omega = 0 \qquad \omega^2 = \frac{\kappa}{m} \qquad \omega^2 = \frac{\kappa}{m}(2m/M + 1) \tag{10}$$

Here we will follow a slightly less systematic approach based on symmetry. First of all there is an obvious zero mode corresponding to a shift of all particles by a finite amount. The potential energy (which depends only on differences) is obviously unchanged by this shift. The zero mode is

$$\boxed{\vec{E}_z = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \omega^2 = 0} \tag{11}$$

Thus a specific solution to the EOM for belonging only zero mode is

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = (A_z t + B_z) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{12}$$

Then we can guess the next two modes based on the symmetry of the problem. The modes should be even (see figure below) and odd (see figure below) under the reflection symmetry of the problem. The even mode is

$$\boxed{\vec{E}_e = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}} \tag{13}$$

and is clearly orthogonal to the zero mode. This means, more physically, that this oscillation does not displace the center of mass. The odd mode takes the form

$$\boxed{\vec{E}_o = \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix}} \tag{14}$$

We require that it be orthogonal to the zero mode,

$$\vec{E}_o^T \cdot \mathcal{M} \cdot \vec{E}_z = m + M\alpha + m = 0\,, \tag{15}$$

or equivalently we the odd mode should not displace the center of mass. Thus

$$\boxed{\alpha = -\frac{2m}{M}} \tag{16}$$

Now we can substitute these two modes into the the equation of motion and determine the frequencies. First we study the even mode, substituting this form into the equations of motion, taking

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = A_e e^{-i\omega_e t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\,, \tag{17}$$

we find

$$-m\omega_e^2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \kappa \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{18}$$

which satisfies the provided

$$\boxed{\omega_e^2 = \frac{\kappa}{m}} \tag{19}$$

Similarly we substitute

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = A_o e^{-i\omega_e t} \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}\,, \tag{20}$$

into the EOM (Eq. (5)), leading to

$$m\omega_o^2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = k(1 + 2m/M) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{21}$$

Thus the EOM are satisfied provided

$$\boxed{\omega_o^2 = \frac{k}{m}\,(1 + 2m/M)} \tag{22}$$

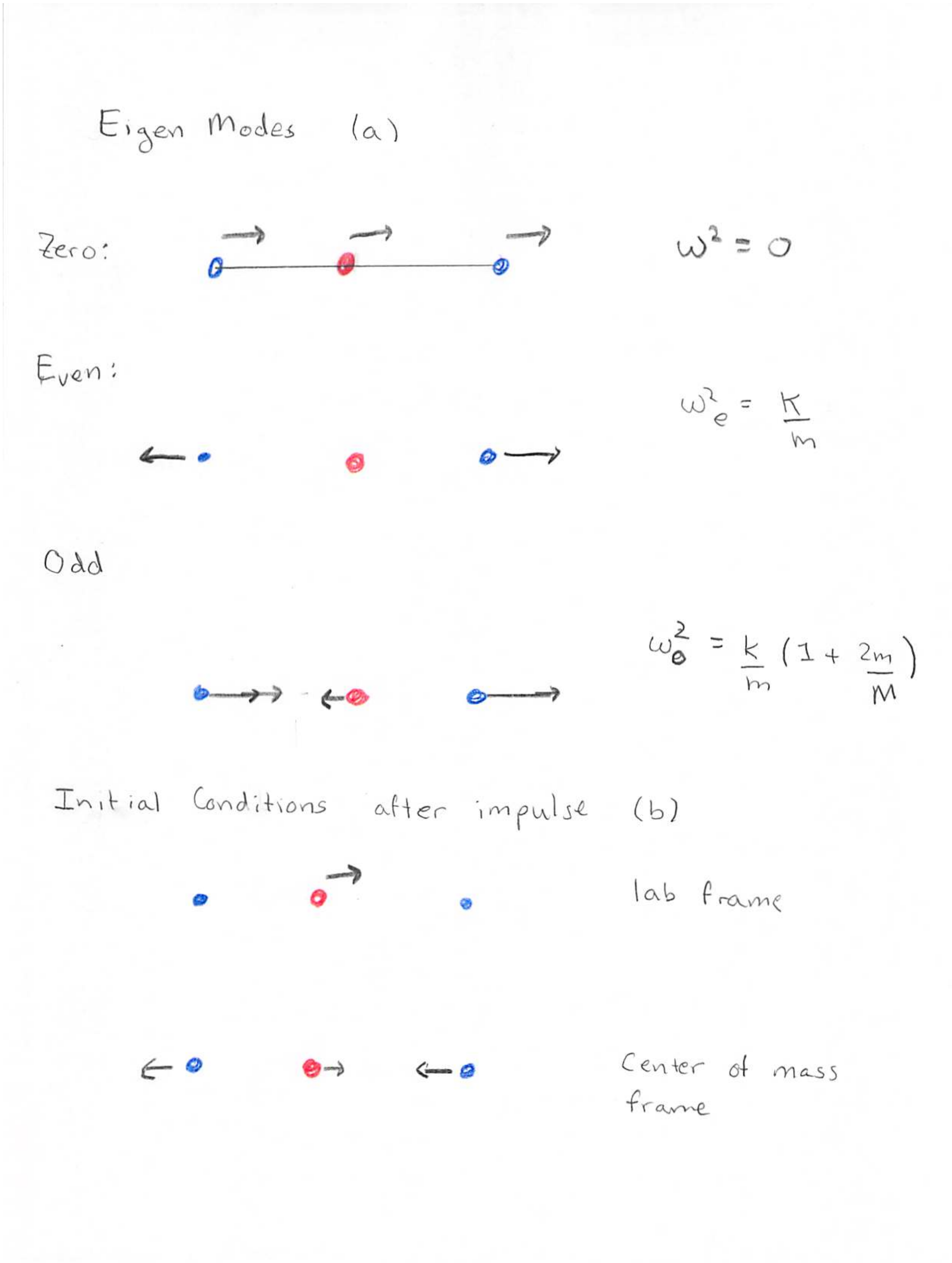


Figure 1: (a) A sketch of the normal modes. (b) The initial conditions after the impulse

(b) Here we need to analyze the motion just after the impulse. Just after the impulse, on the middle particle has any net velocity. Indeed the velocity of the middle particle just after the impulse is simply P_0/M and the velocity of the center of mass is $P_0/(M+2m)$, and the positions of the particles are unchanged (see figur!). Thus the initial conditions for our mechanics problem are

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_X \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ P_0/M \\ 0 \end{pmatrix} \tag{23}$$

The general solution is

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = (A_z t + B_z)\vec{E}_z + A_e \cos(\omega_e t + \varphi_e)\vec{E}_e + A_o \cos(\omega_o t + \varphi_o)\vec{E}_o \tag{24}$$

In general we should adjust the integration constants to match the initial conditions. Straightforward algebra can determine these constants.

To carry out this algebra it helps to have intuition. First, since no particle is displaced at $t = 0_+$ the solution must take the form

$$\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = A_z t\vec{E}_z + A_e \sin(\omega_e t)\vec{E}_e + A_o \sin(\omega_o t)\vec{E}_o \tag{25}$$

This already simplifies the algebra. It is simple enough to stop thinking here and to solve for A_z, A_e, A_o in a pedestrian fashion.

But to clarify the physics we note that after the impulse, the center of mass moves with velocity $v_{cm} = P_0/(M+2m)$. This center of mass motion determines the zero mode, fixing $A_z = v_{\rm cm}$, and $B_z = 0$. Just after the impulse the mass M is moving with speed $v_0 = P_0/M$. In the center of mass frame (see figure) the particles are moving (just after the impulse) with velocity

$$\begin{pmatrix} -v_{\rm cm} \\ v_0 - v_{\rm cm} \\ -v_{\rm cm} \end{pmatrix} \tag{26}$$

which clearly belongs to the odd mode. We may therefore the set the amplitude of the even mode to zero. Then finally we may fix the amplitude of the odd from our initial conditions in the center of mass frame:

$$\begin{pmatrix} -v_{\rm cm} \\ v_0 - v_{\rm cm} \\ -v_{\rm cm} \end{pmatrix} = \frac{d}{dt}(A_0 \sin(\omega_o t))\Big|_{t=0_+} \begin{pmatrix} 1 \\ \frac{-2m}{M} \\ 1 \end{pmatrix} . \tag{27}$$

So we require $-A_o\omega_o = v_{\rm cm}$ leading to our final result

$$\boxed{\begin{pmatrix} x_1 \\ X \\ x_2 \end{pmatrix} = \frac{P_0 t}{M+2m}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{P_0}{(M+2m)\omega_0}\sin(\omega_o t)\begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix} .} \tag{28}$$

Final 2019

3 Torque on a box

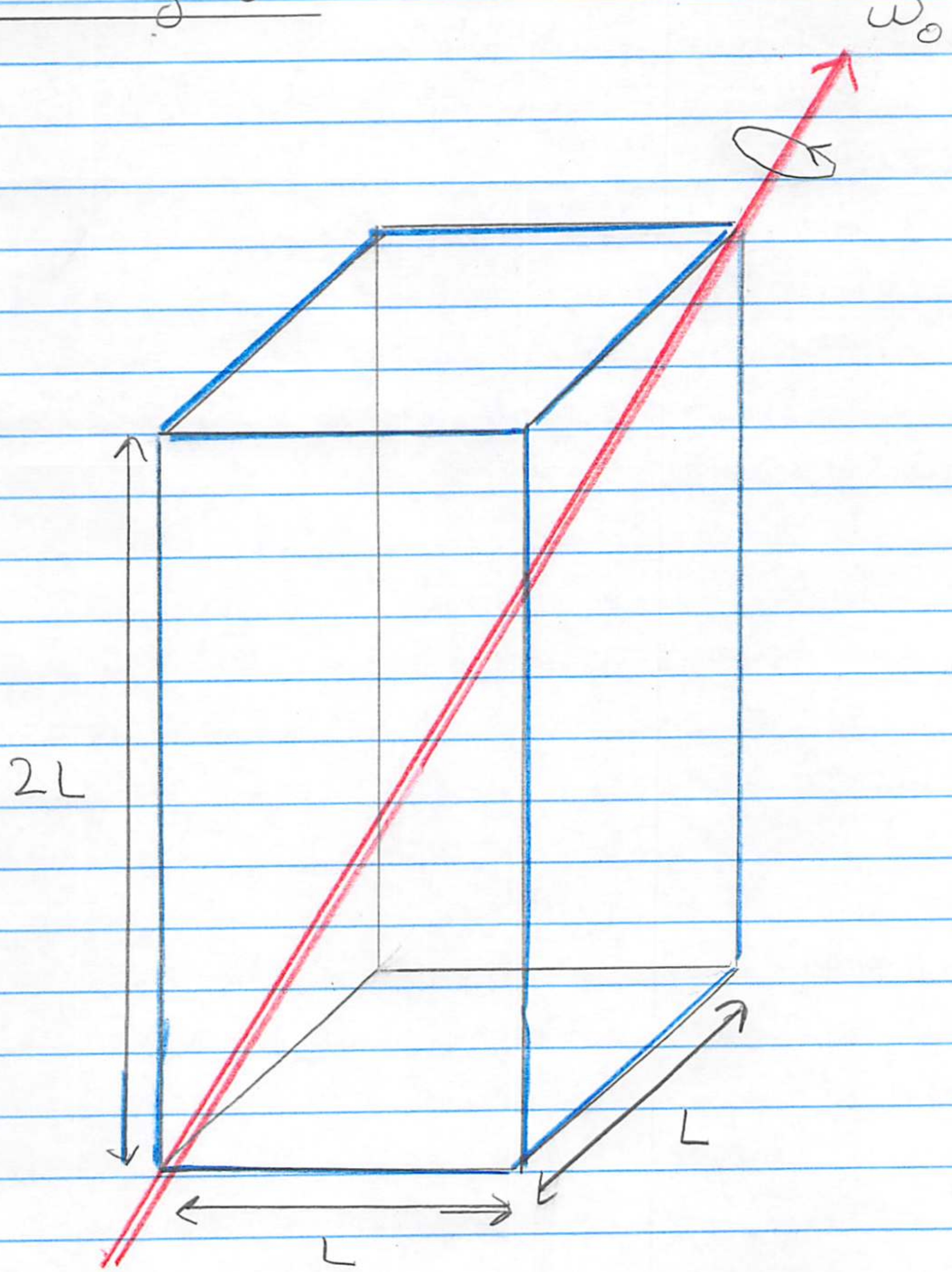
Consider a solid box of mass m and dimension $L, L, 2L$ (see figure).

(a) Compute all components of the moment of inertia tensor around center of mass.

(b) The box is rotated with constant angular frequency ω_0 around its diagonal. At $t = 0$ the box is oriented as shown in the figure. Compute the angular momentum and kinetic energy at this time.

(c) Compute the torque required (both magnitude and direction) to maintain the box's rotational motion at time $t = 0$. Does the magnitude depend on time?

A Rotating Box



Solution

(a) The principal axes are clearly the x, y, z coordinate system

$$I_{xx} = \int dm(y^2 + z^2) \tag{36}$$

$$I_{yy} = \int dm(x^2 + z^2) \tag{37}$$

$$I_{zz} = \int dm(x^2 + y^2) \tag{38}$$

Working through the first example

$$\begin{aligned} I_{xx} =& \frac{m}{2L^3} \times \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \int_{-L}^{L} dz (y^2 + z^2) &(39)\\ =& \frac{m}{2L^3} \left[L \times \left(\frac{2}{3} \frac{L^3}{8} \right) \times 2L + L \times L \times \frac{2}{3} L^3 \right] &(40)\\ =& mL^2 \left[\frac{1}{12} + \frac{1}{3} \right] &(41)\\ =& \frac{5}{12} mL^2 &(42)\end{aligned}$$

The other integrals work out by analogy

$$I_{yy} = I_{xx} \,, \tag{43}$$

while

$$I_{zz} = mL^2 \frac{2}{12} \,. \tag{44}$$

To summarize we have

$$I = \frac{1}{12} mL^2 \begin{pmatrix} 5 & & \\ & 5 & \\ & & 2 \end{pmatrix} \tag{45}$$

(b) Then the angular momentum in the body axes are

$$L_a = I_{ab} \omega_b \tag{46}$$

The angular velocity is $(\omega_1, \omega_2, \omega_3) = \frac{\omega_0}{\sqrt{6}}(1, 1, 2)$. So we find

$$L = \frac{mL^2 \omega_0}{12\sqrt{6}} \begin{pmatrix} 5 & & \\ & 5 & \\ & & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{mL^2 \omega_0}{12\sqrt{6}} \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} \tag{47}$$

The kinetic energy is

$$T = \frac{1}{2} \vec{\omega} \cdot \vec{L} \,, \tag{48}$$

yielding

$$T = \frac{mL^2 \omega_0^2}{2 \cdot 12 \cdot 6} (5 + 5 + 8) \,. \tag{49}$$

(c) The torque is the change in angular momentum

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L} \tag{50}$$

So, up to normalization, we need to compute the cross product of

$$\boldsymbol{v} = (1, 1, 2) \times (5, 5, 4) \propto \vec{\omega} \times \vec{L} \tag{51}$$

Straightforward manipulations shows this is

$$\boldsymbol{v} = (1, 1, 2) \times [(5, 5, 8) - (0, 0, 4)] \,, \tag{52}$$

$$\boldsymbol{v} = (\boldsymbol{e}_x + \boldsymbol{e}_y) \times (-4\boldsymbol{e}_z) \,, \tag{53}$$

$$= 4\boldsymbol{e}_y - 4\boldsymbol{e}_x \,. \tag{54}$$

Inserting the normalization factors we find

$$\vec{\tau} = \frac{mL^2\omega_0}{12\sqrt{6}} \frac{\omega_0}{\sqrt{6}} (4\boldsymbol{e}_y - 4\boldsymbol{e}_x) \tag{55}$$

$$= \frac{mL^2\omega_0^2}{18} (\boldsymbol{e}_y - \boldsymbol{e}_x) \,. \tag{56}$$

The magnitude is indepdent of time

$$\tau = \frac{\sqrt{2}}{18} mL^2\omega_0^2 \,. \tag{57}$$

Final 2018

3 A damped anharmonic oscillator

Consider a damped anharmonic oscillator

$$m\frac{d^2x}{dt} + m\eta\frac{dx}{dt} + m\omega_0^2 x + \lambda x^3 = 0 \tag{43}$$

(a) At time $t = 0$ the oscillator is initialized with initial amplitude, $x(0) = A$, and no initial velocity. (i) Introduce an appropriate set of dimensionless variables to rewrite Eq. (43) in dimensionless form. (ii) For what range of the dimensionful parameters of this problem can the motion of the oscillator be consider approximately harmonic, with small damping and small anharmonic terms?

(b) Determine a zeroth order solution to Eq. (43), which is approximately valid for all times $t > 0$ for small damping and small anharmonic terms.

The identity $(\cos(x))^3 = \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x)$ may be useful.

Solution:

(a) We first set $\omega_0 = A = 1$. Then

$$\hat{x} = \frac{x}{A} \qquad \hat{t} = \omega_0 t \tag{44}$$

The equation then reads

$$m\omega_0^2 A \frac{d^2\hat{x}}{d\hat{t}^2} + (m\omega_0^2 A)\frac{\eta}{\omega_0}\frac{d\hat{x}}{d\hat{t}} + m\omega_0^2 A\hat{x} + \lambda A^3 \hat{x}^3 = 0 \tag{45}$$

After dividing by $m\omega_0^2 A$ we find

$$\frac{d^2\hat{x}}{d\hat{t}^2} + \frac{\eta}{\omega_0}\frac{d\hat{x}}{d\hat{t}} + \hat{x} + \frac{\lambda A^2}{m\omega_0^2}\hat{x}^3 = 0 \tag{46}$$

So we require that

$$\frac{\eta}{\omega_0} \ll 1 \tag{47}$$

and

$$\frac{\lambda A^2}{m\omega_0^2} \ll 1 \tag{48}$$

Finally we define

$$\hat{\eta} = \frac{\eta}{\omega_0} \qquad \hat{\lambda} \equiv \frac{\lambda A^2}{m\omega_0^2} \tag{49}$$

and we will drop the "hats" below in all expressions.

(b) We substitute a trial solution and define the phase Ω

$$x(t) = C(t)\cos(-t + \varphi(t)) \qquad \Omega \equiv -t + \varphi \tag{50}$$

The spirit here is that in the absence of the damping and anharmonic terms the general solution is

$$x(t) = C\cos(-t + \varphi)\,. \tag{51}$$

In secular perturbation theory the integration constants C, φ are promoted to slow functions of time. This approximation scheme goes by various names such as the "slow-roll" approximation, or the "rotating waver approximation".

Computing the derivative

$$\frac{dx}{dt} = -\,C\sin(-t+\varphi)(-t+\dot{\varphi}) + \dot{C}\cos\Omega \tag{52}$$

$$= C\sin(\Omega) - C\sin(\Omega)\dot{\varphi} + \dot{C}\cos\Omega \tag{53}$$

Differentiating again

$$\frac{d^2x}{dt^2} = C\cos(\Omega)(-1+\dot{\varphi}) + \dot{C}\sin\Omega + C\cos(\Omega)\dot{\varphi} + \dot{C}\sin\Omega + \text{small} \tag{54}$$

$$= -\,C\cos\Omega + 2\dot{C}\sin\Omega + 2C\cos\Omega\dot{\varphi} \tag{55}$$

In this step we have neglected terms of order $\ddot{C}$ and $\ddot{\varphi}$. This is the spirit of the "slow-roll" approximation where the integation "constants" depend slowly on time.

Thus

$$\ddot{x} + x = 2\dot{C}\sin\Omega + 2C\cos\Omega\dot{\varphi} \tag{56}$$

The anharmonic term apprixmates to

$$\lambda x^3 = \frac{3}{4}\cos(\Omega) + \frac{1}{4}\cos(3\Omega) \tag{57}$$

$$\simeq \frac{3}{4}\cos(\Omega) \tag{58}$$

while the dissipative term is

$$\eta\frac{dx}{dt} = \eta C\sin(\Omega) \tag{59}$$

In these terms which are small because of the small parametes, λ and η, we may neglect $\dot{C}$ and $\dot{\varphi}$. The 3Ω terms contributes to the first correction, but do not contribute to the leading order solution (see lecture). It will be neglected below.

Collecting the partial results for the second derivate, Eq. (56), the anharmonic equation, Eq. (58), and the dissipative equation, Eq. (59), the equation of motion then reads

$$\left(2\dot{C} + \eta C\right)\sin\Omega + \left(2C\dot{\varphi} + \frac{3}{4}\lambda C^3\right) = 0 \tag{60}$$

Integrating the first equation

$$C(t) = C(0)e^{-(\eta/2)t} \tag{61}$$

At early times our solution must behave as $\cos(-kt)$, where k is a constant to be determined. This is a consequence of our initial conditions, where the initial amplitude is unity and the initial velocity is zero. We thus set $C(0)$ to unity.

Then we integrate the phase

$$\dot{\varphi} = \frac{3\lambda}{8}C^2 = \frac{3}{8}\lambda e^{-\eta t}\,, \tag{62}$$

Integrating the phase we find

$$\varphi = \frac{3\lambda}{8\eta}(1 - e^{-\eta t})\,, \tag{63}$$

In integrating the phase we have specified that at small times the phase is $\varphi \propto t$ as required by our initial conditions that the initial velocity be zero. Summarizing then we find

$$\boxed{x(t) = e^{-(\eta/2)t}\cos\left[-t + \frac{3\lambda}{8\eta}(1 - e^{-\eta t})\right]\,.} \tag{64}$$

Midterm 2018: No solution (sorry)

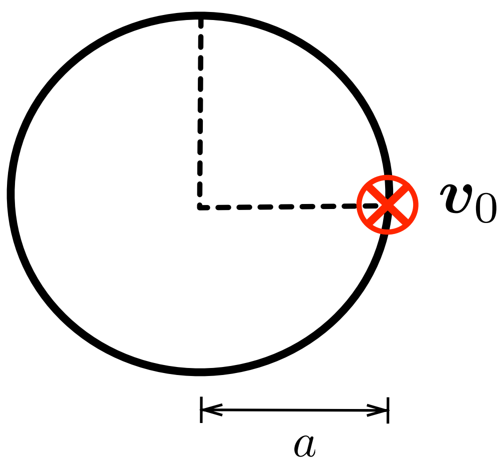
Problem 1. A particle on a sphere

A particle of mass m is confined to move without friction on the surface of a sphere of radius a. The acceleration due to gravity is g and is directed downward.

(a) Using spherical coordinates, determine the Lagrangian and Hamiltonian of the system. Use the Hamiltonian formalism to determine the equations of motion, and to identify an effective potential for θ.

(b) What are the integrals of motion? State the symmetries of the action which lead to the integrals of motion.

(c) At time $t = 0$ the particle is at an angle of $\pi/2$, and is moving with a velocity $\boldsymbol{v}_0$ in the azimuthal direction (into the page):



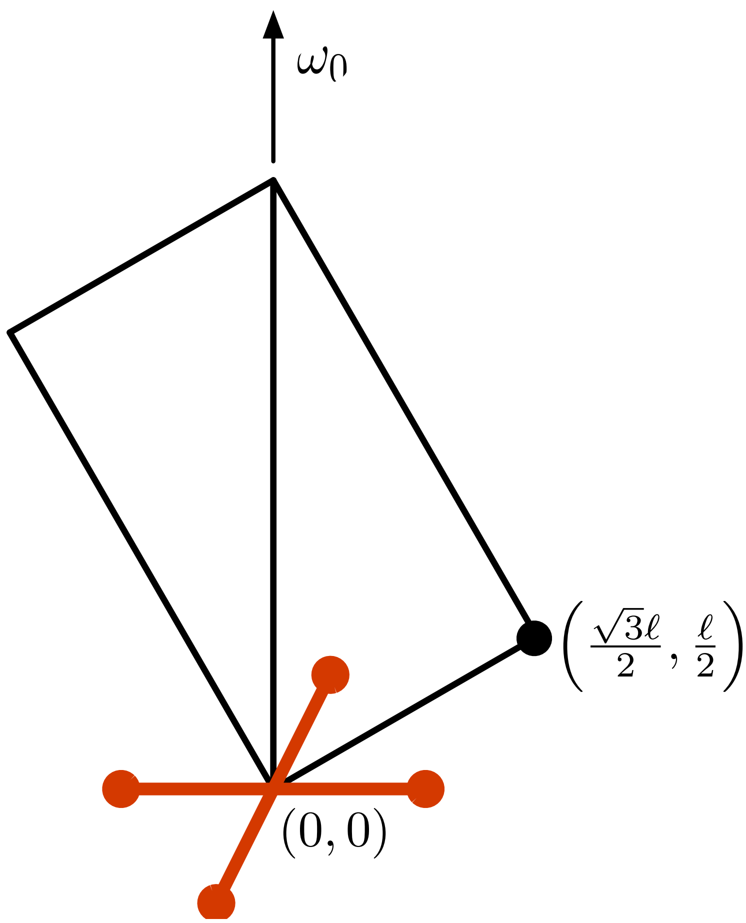
(i) What is the condition on v_0 so that gravity may be treated as a perturbation on the subseqent motion of the particle. What is the trajectory of the particle in the absence of gravity?

(ii) Treating gravity as a small perturbation, what is the amplitude of the subsequent oscillations in θ for the initial conditions described above.

Midterm 2018: No solution (sorry)

Problem 2. A rectangle on a stand

A rectangle of width ℓ and length $\sqrt{3}\ell$ consists of a light frame and a ball of mass m at one corner. The (x, y) coordinates of of the corner with the ball at $t = 0$ is given in the figure below. The rectangle is rotating around its diagonal (the y axis) at frequency ω_0. The rectangle is supported by a stand with four feet at a distance a from the central shaft. At time $t = 0$ the plane of the rectangle lies in the x, y plane.



(a) Compute the moment of inertia tensor at $t = 0$ (around the origin). Use this tensor to find the angular momentum around the origin as a function of time, and check your result using $\boldsymbol{r} \times \boldsymbol{p}$.

(b) Determine the normal force on all four pins at time $t = 0$.