Physics 501: Classical Mechanics

Midterm Exam

Stony Brook University

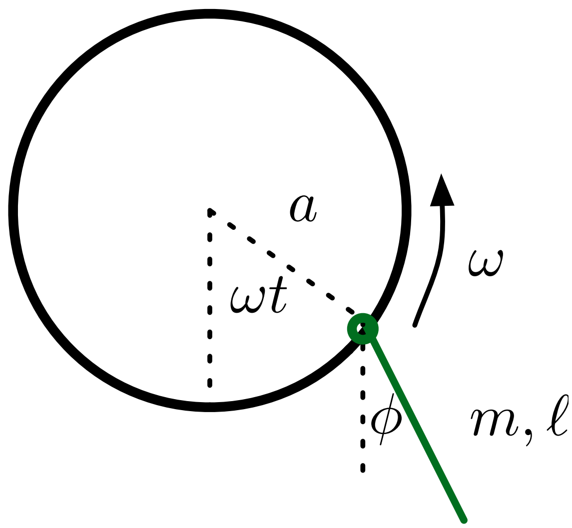
Fall 2020

General Instructions:

You may use one page (front and back) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Problem 1. A pendulum on a wheel

Consider a pendulum consisting of a uniform rod of length ℓ and mass m in the earth's gravitational field. The pivot point of the pendulum is attached to the rim of a wheel of radius a which turns with angular velocity ω, but otherwise the angle of the pendulum is able to rotate freely around its pivot point.



(a) Determine the Lagrangian of the system. You should find after suitable manipulations that the Lagrangian can be written

$$L = \frac{1}{6} m\ell^2 \dot{\phi}^2 + \frac{1}{2} ma\ell\omega^2 \cos(\omega t - \phi) + \frac{1}{2} mg\ell \cos\phi \tag{1}$$

You will be graded on the clarity of the derivation not the answer.

(b) Describe the motion qualitatively when ω is small, and when ω is large? Define what is meant by fast and slow in this context. When ω is arbitrarily fast determine the determine the steady state value of ϕ as a function of time.

(c) When ω is fast, but not arbitrarily fast, the steady state of part (b) will be perturbed by gravity, and ϕ will oscillate around its steady state value. Expand the Lagrangian to quadratic order in $\delta\phi$, and find the resulting equation of motion for $\delta\phi$.

(d) Determine the steady state amplitude and frequency of the resulting oscillations to lowest non-trivial order in the gravitational perturbation, assuming that you are far from any resonance.

(e) If the rotational frequency is twice the resonant frequency of the oscillator of part (c), the perturbative expansion developed in (d) will break down. Explain why.

(a) The position of any point u along the rod (with $u = 0 \ldots \ell$) is

$$x = a\sin(\omega t) + u\sin(\phi) \tag{2}$$
$$y = -a\cos(\omega t) - u\cos(\phi) \tag{3}$$

So the velocity is

$$\dot{x} = a\omega\cos(\omega t) + u\cos\phi\,\dot{\phi} \tag{4}$$
$$\dot{y} = a\omega\sin(\omega t) + u\sin\phi\,\dot{\phi} \tag{5}$$

Thus the kinetic energy is

$$T = \frac{1}{2}\int dm(\dot{x}^2 + \dot{y}^2) \tag{6}$$
$$= \frac{1}{2}\int dm\left(u^2\dot{\phi}^2 + 2au\omega\dot{\phi}\left[\cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi\right] + \text{totderiv}\right) \tag{7}$$
$$= \frac{1}{2}\int dm\left(u^2\dot{\phi}^2 + 2au\omega\dot{\phi}\cos(\omega t - \phi)\right) \tag{8}$$
$$= \frac{1}{2}\int dm\left(u^2\dot{\phi}^2 + 2au\omega\left[-\frac{d}{dt}(\sin(\omega t - \phi)) + \omega\cos(\omega t - \phi)\right]\right) \tag{9}$$
$$= \frac{1}{2}\int dm\left(u^2\dot{\phi}^2 + 2au\omega^2\cos(\omega t - \phi) + \text{totderiv}\right) \tag{10}$$
$$= \frac{1}{6}m\ell^2\dot{\phi}^2 + \frac{1}{2}ma\ell\omega^2\cos(\omega t - \phi) \tag{11}$$

The potential energy is

$$U = \int dmgy = -\frac{1}{2}\left(\frac{mg}{\ell}\right)\ell^2\cos\phi + \text{totderiv} \tag{12}$$

Thus the Lagrangian is

$$L = \frac{1}{6}m\ell^2\dot{\phi}^2 + \frac{1}{2}ma\ell\omega^2\cos(\omega t - \phi) + \frac{1}{2}mg\ell\cos\phi \tag{13}$$

(b) Clearly if ω is small, the pendulum just makes small oscillations. If ω is fast the centripetal force (in the a rotating frame picture) is extremely large, and the pendulum is pushed outward from the center as far as possible. In this limit the steady state configuration is $\phi = \omega t$. Small and fast mean that we should compare the two terms in the Lagrangian. The frequency is fast when

$$ma\ell\omega^2 \gg mg\ell \tag{14}$$

Then we have

$$\omega \gg \sqrt{\frac{g}{a}} \tag{15}$$

(c) Then we write $\phi = \omega t + \psi$ with the understanding that $\psi \equiv \delta\phi$ is small. The Lagrangian is

$$
\begin{aligned}
L =& \frac{1}{6} m\ell^2 (\omega + \dot{\psi})^2 + \frac{1}{2} m a \ell \omega^2 \cos(\psi) + \frac{1}{2} m g \ell \cos(\omega t + \psi) && (16) \\
=& \frac{1}{6} m\ell^2 \dot{\psi}^2 + \frac{1}{6} 2 m\ell^2 \omega \dot{\psi} + \frac{1}{2} m a \ell \omega^2 \cos(\psi) - \frac{1}{2} m g \ell \cos(\omega t + \psi) + \text{const} && (17) \\
=& \frac{1}{6} m\ell^2 \dot{\psi}^2 + \frac{1}{2} m a \ell \omega^2 \cos(\psi) + \frac{1}{2} m g \ell \cos(\omega t + \psi) + \text{totderivs} && (18) \\
=& \frac{1}{6} m\ell^2 \dot{\psi}^2 + \frac{1}{2} m a \ell \omega^2 \cos(\psi) + \frac{1}{2} m g \ell \left(\cos(\omega t) \cos\psi - \sin\omega t \sin(\psi) \right) && (19) \\
\simeq& \frac{1}{6} m\ell^2 \dot{\psi}^2 - \frac{1}{4} m a \ell \omega^2 \left(1 + \frac{g}{a\omega^2} \cos(\omega t) \right) \psi^2 - \frac{1}{2} m g \ell \sin(\omega t) \psi && (20)
\end{aligned}
$$

The equation of motion

$$
\frac{1}{3} m\ell^2 \ddot{\psi} + \frac{1}{2} m a \ell \omega^2 \left(1 + (g/a\omega^2) \cos(\omega t) \right) \psi = -\frac{1}{2} m g \ell \sin(\omega t) \tag{21}
$$

So dividing by $1/3 m\ell^2$ the equation of motion takes the form

$$
\ddot{\psi} + \omega_0^2 (1 + \epsilon \cos(\omega t)) \psi = -\Omega^2 \sin(\omega t) \tag{22}
$$

Then

$$
\begin{aligned}
\omega_0^2 =& \frac{3}{2} \frac{a}{\ell} \omega^2 && (23) \\
\epsilon =& \frac{g}{a\omega^2} && (24) \\
\Omega^2 =& \frac{3}{2} \frac{g}{\ell} && (25)
\end{aligned}
$$

(d) At lowest order $\Omega^2 \ll \omega^2$, we drop terms of order ϵ. We try $\psi = A \sin(\omega t)$ and find

$$
\psi^{(0)} = \frac{-\Omega^2 \sin(\omega t)}{-\omega^2 + \omega_0^2} \tag{26}
$$

(e) When keeping the term $\omega_0^2 = \epsilon \cos(\omega t) \psi$ the problem takes the form of a parametric oscillator where

$$
\omega_0^2(t) = \omega_0^2 (1 + \epsilon \cos(\omega t)) \tag{27}
$$

As analyzed in class when ω is twice ω_0 then a parametric oscillator experiences runaway growth for arbitrarily small ϵ. Requiring $\omega = 2\omega_0$ means $\ell = 6a$.

Problem 2. General variation

Consider the action

$$S[q(t); q_1, t_1, q_2 t_2] = \int_{t_1}^{t_2} dt L(q, \dot{q}, t) \tag{28}$$

evaluated on a trajectory $q(t)$ with $t \in [t_1, t_2]$ with endpoints $q(t_1) = q_1$ and $q(t_2) = q_2$. Determine how the action is changed by a general infinitessimal variation

$$q(t) \to q + \delta q(t) \tag{29}$$

$$t_1 \to t_1 + \delta t_1 \tag{30}$$

$$t_2 \to t_2 + \delta t_2 \tag{31}$$

Assume that $q(t)$ satisfies the equations of motion, but do not assume that $\delta q(t)$ vanishes at the end points. Explain all of your steps.

Solution

First let's guess the answer. Since

$$S = \int p dq - H dt \tag{32}$$

We anticipate that

$$\delta S = p_2 \delta q_2 - p_1 \delta q_1 - H_2 \delta t_2 + H_1 \delta t_1 \tag{33}$$

The analysis below in the Lagrangian framework confirms this.

The new path is $\tilde{q}(t)$ with $t \in [t_1 + \delta t_1, t_2 + \delta t_2]$. Then

$$\tilde{q}(t_2 + \delta t_2) \equiv q_2 + \delta q_2 \tag{34}$$

$$\tilde{q}(t_1 + \delta t_1) \equiv q_1 + \delta q_1 \tag{35}$$

Then note (see picture) that the variation at the original endpoints involves the extrapolation

$$\Delta q_2 \equiv \tilde{q}(t_2) - q(t_2) = \delta q_2 - \dot{q}(t_2)\delta t_2 \tag{36}$$

$$\Delta q_1 \equiv \tilde{q}(t_1) - q(t_1) = \delta q_1 - \dot{q}(t_1)\delta t_1 \tag{37}$$

or if you prefer:

$$\Delta q_2 + \dot{q}(t_2)\delta t_2 = \delta q_1 \tag{38}$$

$$\Delta q_1 + \dot{q}(t_1)\delta t_1 = \delta q_2 \tag{39}$$

So

$$S = \int_{t_1+\delta t_1}^{t_2+\delta t_2} dt L(\tilde{q}, \dot{\tilde{q}}, t) \tag{40}$$

$$\simeq S^{(0)} + L_2 \delta t_2 - L_1 \delta t_1 + \int_{t_1}^{t_2} dt L(\tilde{q}, \dot{\tilde{q}}, t) \tag{41}$$

Now we do the usual steps writing $\tilde{q} = q + \delta q$, integrating by parts etc. , and find

$$S \simeq S^{(0)} + L_2 \delta t_2 - L_1 \delta t_1 + \frac{\partial L}{\partial \dot{q}} \delta q(t) \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt' \underbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right)}_{=0} \delta q \tag{42}$$

Using the EOM and the definition of $\Delta q = \tilde{q}(t) - q(t)$ given in Eq. (38) we have

$$S \simeq S^{(0)} + L_2 \delta t_2 - L_1 \delta t_1 + p_2 \Delta q_2 - p_1 \Delta q_1 \tag{43}$$

We have finally using the extrapoltations in Eq. (38)

$$\delta S = (p_2 \delta q_2 - H_2 \delta t_2) - (p_1 \delta q_1 - H_1 \delta t_1) \tag{44}$$

where $H = p\dot{q} - L$.

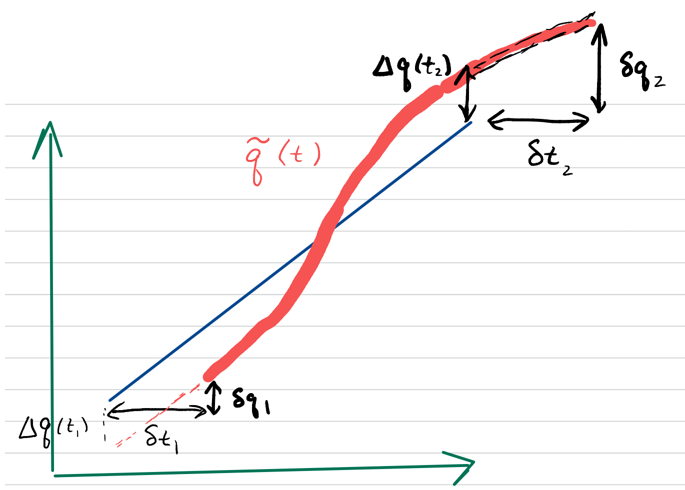


Figure 1: A general variation. Note that $\Delta q_2 + \dot{q}(t_2)\delta t_2 = \delta q_2$ and analogously for q_1.

Problem 3. A curious Lagrangian

Consider a Lagrangian

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega_0^2 q^2 + \frac{\kappa}{4}\dot{q}^4 \tag{45}$$

(a) The motion is started with initial conditions

$$q(0) = x_0 \qquad \dot{q}(0) = 0 \tag{46}$$

Under what conditions can the term $\kappa\dot{q}^4/4$ be considered a small perturbation?

(b) Determine the motion system treating $\kappa\dot{q}^4/4$ as perturbation, and working to 1st order in secular perturbation theory.

Solution

(a) We should compare the two terms in the Lagrangian. Estimating $\dot{q} \sim x_0\omega_0$ we find

$$\kappa x_0^4 \omega_0^4 \ll m\omega_0^2 x_0^2 \tag{47}$$

So we require that

$$\frac{\kappa}{m}\omega_0^2 x_0^2 \ll 1 \tag{48}$$

(b) The equation of motion is

$$\frac{d}{dt}\left(m\dot{q} + \kappa\dot{q}^3\right) = -m\omega_0^2 q \tag{49}$$

Or

$$\ddot{q} + \omega_0^2 q = -\frac{\kappa}{m}\frac{d}{dt}\dot{q}^3 \tag{50}$$

So we make the ansatz

$$q(t) = q^{(0)} + q^{(1)}\,, \tag{51}$$

where the zeroth order solution is

$$q^{(0)}(t) = A(t)\cos\Psi \qquad \Psi(t) = -\omega_0 t + \varphi(t)\,. \tag{52}$$

With the usual secular (slow roll) approximations we have

$$\ddot{q}^{(0)} + \omega_0^2 q^{(0)} \simeq 2\omega_0 A\dot{\varphi}\cos(\Psi) + 2\omega_0\dot{A}\sin(\Psi)\,. \tag{53}$$

The peturbing tem is can be treated at lowest possible order in the slow roll approximation where A and φ are treated as constants:

$$-\frac{\kappa}{m}\partial_t\dot{q}^3 \simeq \frac{\kappa}{m}\frac{d}{dt}(\omega_0\sin(\Psi))^3 \simeq 3A^3\omega_0^4\,\frac{\kappa}{m}\sin(\Psi)^2\cos(\Psi)\,. \tag{54}$$

We can resolve this in fourier components:

$$\sin(\Psi)^2\cos(\Psi) = \underbrace{\frac{1}{4}\cos\Psi}_{\text{on resonance/0th order}} \quad -\underbrace{\frac{1}{4}\cos(3\Psi)}_{\text{1st}} \tag{55}$$

Putting together the ingrediants

$$\left(\ddot{q}^{(0)} + \omega_0^2 q^{(0)}\right) + \left(\ddot{q}^{(1)} + \omega_0^2 q^{(1)}\right) = 3A^3\omega_0^4\,\frac{\kappa}{m}\Bigg(\underbrace{\frac{1}{4}\cos\Psi}_{\text{on resonance/0th order}} \quad -\underbrace{\frac{1}{4}\cos(3\Psi)}_{\text{1st}}\Bigg) \tag{56}$$

So comparing the zeroth order terms in this expression

$$2\omega_0 A\dot{\varphi}\cos(\Psi) + 2\omega_0\dot{A}\sin(\Psi) = 3A^3\omega_0^4\,\frac{\kappa}{m}\frac{1}{4}\cos\Psi\,. \tag{57}$$

Or

$$\dot{A} = 0 \qquad 2\omega_0 A\dot{\varphi} = \frac{3}{4}A^3\omega_0^4\frac{\kappa}{m} \tag{58}$$

So we find

$$\varphi = \left(\frac{3}{8}A^2\omega_0^2\frac{\kappa}{m}\right)\omega_0 t \tag{59}$$

This is a simple frequency shift

$$\Psi = -\omega t \qquad \omega = \omega_0\left(1 - \frac{3}{8}A^2\omega_0^2\frac{\kappa}{m}\right) \tag{60}$$

For the first order terms we solve for the steady state oscillations of the driven harmonic oscillator

$$\left(\ddot{q}^{(1)} + \omega_0^2 q^{(1)}\right) = 3A^3\omega_0^4\,\frac{\kappa}{m}\Big(-\underbrace{\frac{1}{4}\cos(3\Psi)}_{\text{1st}}\Big) \tag{61}$$

And find

$$\begin{aligned} q^{(1)} =& \frac{-\frac{3}{4}\frac{\kappa}{m}A^3\omega_0^4}{-(3\omega_0)^2 + \omega_0^2}\cos(3\Psi) && (62)\\ =& \frac{3}{32}\frac{\kappa}{m}A^2\omega_0^2\,A\cos(3\Psi) && (63)\end{aligned}$$

To summarize we have

$$q(t) \simeq A\cos(\Psi) + \left(\frac{3}{32}\frac{\kappa}{m}A^2\omega_0^2\right)A\cos(3\Psi) \tag{64}$$

with

$$\Psi = -\omega t \qquad \omega = \omega_0\left(1 - \frac{3}{8}A^2\omega_0^2\frac{\kappa}{m}\right) \tag{65}$$

We have from the initial conditions

$$x_0 = A + \left(\frac{3}{32}\frac{\kappa}{m}A^2\omega_0^2\right)A \tag{66}$$

which is solved iteratively

$$A \simeq x_0 - \left(\frac{3}{32}\frac{\kappa}{m}x_0^2\omega_0^2\right)x_0 \tag{67}$$

Leading to a final result

$$q(t) \simeq \left(1 - \left(\frac{3}{32}\frac{\kappa}{m}x_0^2\omega_0^2\right)\right)x_0\cos(\Psi) + \left(\frac{3}{32}\frac{\kappa}{m}x_0^2\omega_0^2\right)\cos(3\Psi) \tag{68}$$