Forced Oscillations

(Likharev, EGP, section 4.1)

- Consider the damped oscillator

$$\underbrace{\left(m\frac{d^2}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2 \right)}_{\text{definition } \equiv \mathcal{L}_t \text{ = a linear operator}} x = F(t)$$

- And the EOM is $\mathcal{L}_t\, x(t) = F(t)$.

- The general solution is a specific solution $x_s(t)$ plus a homogeneous solution, $x_h(t)$

$$\mathcal{L}_t\, x_s(t) = F(t)$$

$$\mathcal{L}_t\, x_h(t) = 0$$

And

$$x(t) = x_s(t) + x_h(t)$$

- The homogeneous solution is found by substituting $x(t) = Ae^{-i\omega t}$ into the EOM. Each d/dt gets replaced by $-i\omega$, $d/dt \to -i\omega$

$$\underbrace{(-m\omega^2 - m\eta i\omega + m\omega_0^2)}_{\text{Set this to zero and solve for } \omega_\pm} Ae^{-i\omega t} = 0$$

- The two characteristic roots are:

$$\omega_\pm = \pm\left(\omega_0^2 - \left(\frac{\eta}{2}\right)^2\right)^{1/2} - i\frac{\eta}{2} \equiv \pm\,\omega' - i\frac{\eta}{2} \qquad \boxed{(\omega')^2 \equiv \omega_0^2 - \left(\frac{\eta}{2}\right)^2}$$

$$\boxed{\omega_\pm \simeq \pm\,\omega_0 - i\eta/2}$$ for damping small

- The general solution is

$$x_h(t) = A e^{-i\omega_+ t} + B e^{-i\omega_- t}$$

Now x_h is real so we can limit ourselves to one of these exponentals:

$$x_h(t) = \mathrm{Re}\left[A e^{-i\omega_+ t}\right] \qquad A = |A| e^{i\phi}$$

$$= \boxed{|A| \cos(-\omega' t + \phi)\, e^{-\eta/2\, t}}$$

Response: Consider the response to a force:

$$F(t) = \mathrm{Re}\left(F_\omega e^{-i\omega t}\right) = F_\omega \cos(\omega t)$$

Then we may solve for the specific solution:

$$\left(m\frac{d^2 x}{dt^2} + m\eta\frac{d}{dt} + m\omega_0^2\right) x_s(t) = F_\omega e^{-i\omega t}$$

- Solving by guessing $x_s(t) = x_\omega e^{-i\omega t}$ we find:

$$m(-\omega^2 + \omega_0^2 - i\omega\eta)\, x_\omega = F_\omega$$

Or

$$x_\omega = G_R(\omega)\, F_\omega \qquad \boxed{G_R(\omega) \equiv \frac{1/m}{[-\omega^2+\omega_0^2 - i\omega\eta]}}$$

- $G_R(\omega)$ is known as the (retarded) responce function. It is the ratio between the sinusoidal signal and the driving force. We will discuss it further below. For now, the specific solution is:

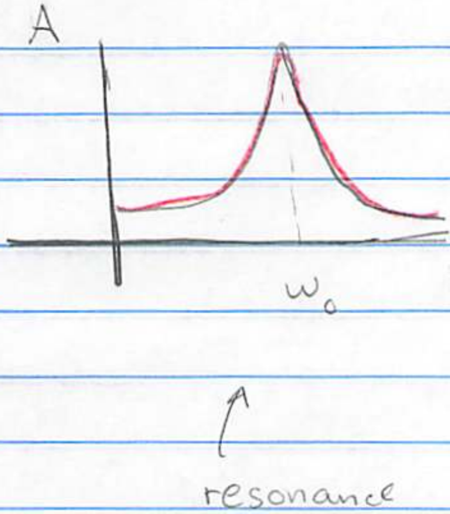
$$x_s(t) = \mathrm{Re}\, x_\omega e^{-i\omega t} = A\cos(-\omega t + \phi)$$

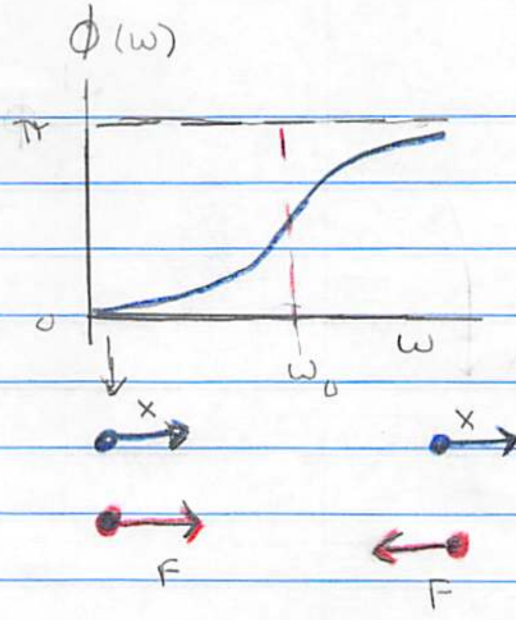
with

$$A = |x_\omega| = F_\omega |G_R(\omega)|$$

$$\phi = \text{phase of } x_\omega = \tan^{-1}\frac{\mathrm{Im}\, G_R}{\mathrm{Re}\, G_R}$$

Shows a characteristic resonance structure





- x and F are 180° out of phase at high ω
- x and F are in phase at low ω

Resonant Behavior

(Landau: Section 22)

- Now lets consider the special case of zero damping and on resonance. Our formalism doesn't work, since x_ω diverges for $\omega \to \omega_0$

$$x_\omega = \frac{F_\omega / m}{-\omega^2 + \omega_0^2}$$

- Let us study the limit $\omega \to \omega_0$. Our general solution is

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t + \frac{F_\omega/m}{-\omega^2+\omega_0^2}\cos\omega t$$

- Regrouping with a new $A \to A'$

$$x(t) = A'\cos\omega_0 t + B\sin\omega_0 t + F_\omega/m\left[\frac{\cos\omega t - \cos\omega_0 t}{-\omega^2+\omega_0^2}\right]$$

- writing $\omega = \omega_0 + \Delta\omega$ and expanding in $\Delta\omega$ to 1st order

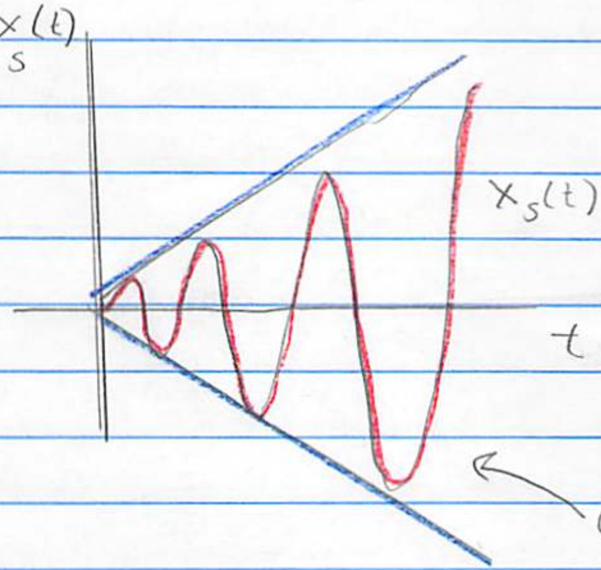
$$\cos\omega t = \cos\omega_0 t \cos\Delta\omega t - \sin\omega_0 t \sin\Delta\omega t$$

$$\simeq \cos\omega_0 t - \sin(\omega_0 t)\, \Delta\omega t$$

- Yielding

$$x(t) = \underbrace{A' \cos\omega_0 t + B \sin\omega_0 t}_{x_h(t)} + \underbrace{\frac{F_0}{2m\omega_0} t \sin\omega_0 t}_{x_s(t)}$$

$x_s(t)$ grows in time!



Called a secular divergence, meaning grows with time

- This will be important for non-linear oscillations