

## The Hamiltonian

- We would like to formulate classical mechanics in terms of coordinates  $q_A$  and momenta  $p_A$ , instead of coordinates  $q_A$  and velocities,  $v_A \equiv \dot{q}_A$ . We will make a Legendre transformation, which we describe below

• First; note

$$dL = \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial t} dt$$

$$dL = p dV + \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial t} dt$$

$\underbrace{\hspace{10em}}_{\equiv \text{spectators}}$

these variables are not essential to the transformation

- So consider  $dH = d(pv - L)$

$$d(pv - L) = v dp + p dV - (p dV + \text{spectators})$$

So

Important

★

$$dH = v dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt$$

In general if  $q, v, t$  are our independent variables we would write:

$$dp = \frac{\partial p}{\partial q} dq + \frac{\partial p}{\partial v} dv$$

• But since  $dp$  appears naturally it is useful/insightful to take  $p, q, t$  instead of  $v, q, t$  as the independent variables

• We are allowed to replace  $v$  with  $p$  provided for each  $v$  there is one value of  $p$ . I.e. given  $p$  as a function of  $v$ ,  $p(v) = \frac{\partial L}{\partial v}$ ,

we can invert this equation (at each time and coordinate  $q$ ) to find  $v$  as a function of  $p$ ,  $v(p)$

• So then  $H$  as a function of  $p, q, t$  is

$$\underline{H(p, q) = p v(p) - L(q, v(p), t)}$$

★ Example:  $L = \frac{1}{2} m v^2 - u(x)$ ,  $p = \partial L / \partial v = mv$

Inverting  $p = mv$  we have  $v = p/m$  and thus

$$\begin{aligned} H &= p \left( \frac{p}{m} \right) - \left( \frac{1}{2} m \left( \frac{p}{m} \right)^2 - u(x) \right) - \\ &= \frac{p^2}{2m} + u(q) \end{aligned}$$

- Now once we have taken  $p, q, t$  as the independent variables, since (see above)

$$\star dH = v dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt \quad \left. \vphantom{dH} \right\} \text{Implies also:}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad \frac{\partial H}{\partial q} = \frac{\partial L}{\partial q}$$

So

$$\boxed{\frac{\partial H}{\partial p} = v} \Rightarrow \boxed{\frac{dq}{dt} = \frac{\partial H}{\partial p}}$$

Hamilton  
Equations

And from the Lagrange equation

$$\frac{d}{dt} p = \frac{\partial L}{\partial q} \Rightarrow \boxed{\frac{dp}{dt} = -\frac{\partial H}{\partial q}}$$

- So the Hamiltonian can be used to determine the evolution of the system. The equations above are Hamilton's Eom and are equivalent to the Euler-Lagrange equations:

For the example:

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m} \checkmark$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x} \checkmark$$

\* The hamiltonian is constant in time  
if  $H$  is not an explicit function of time

$\dot{H} =$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial p} \frac{dp}{dt} + \frac{\partial H}{\partial t}$$



these cancel by the EOM

$$= \frac{\partial H}{\partial t}$$

The Hamiltonian for an important Lagrangian

• This is related to a particle in a magnetic field

$$L = \frac{1}{2} a_{ij}(q) \dot{q}^i \dot{q}^j + b_i \dot{q}^i - U(q)$$

↑ symmetric

$a, b, U$  are functions of  $q$

Then

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = a_{ij} \dot{q}^j + b_i$$

Inverting to express  $\dot{q}$  in terms of  $p$

$$\star \dot{q}^i = (a^{-1})^{ij} (p_j - b_j)$$

$a^{-1}$  inverse matrix,  
i.e.  $a^{-1}a = \mathbb{1}$

So

$$\begin{aligned} H = p_i \dot{q}^i - L &= p_i \dot{q}^i - \left( \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + b_i \dot{q}^i - U \right) \\ &= (p_i - b_i) \dot{q}^i - \left( \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j - U \right) \end{aligned}$$

Using  $\star$  we have after minor algebra

$$H = \frac{1}{2} (a^{-1})^{ij} (p_i - b_i) (p_j - b_j) + U$$

Kinetic energy

• Remark:

Some of you might prefer a matrix notation:

$$Q = \begin{pmatrix} q^1 \\ \vdots \\ q^n \end{pmatrix}$$

$$P = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

But try to get over this hangup!

$$A = \begin{pmatrix} a_{ij} \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

• Then

$$L = \frac{1}{2} \dot{Q}^T A \dot{Q} + B^T \dot{Q} - U$$

$$\frac{\partial L}{\partial \dot{Q}} = P = A \dot{Q} + B \implies \star \dot{Q} = A^{-1} (P - B)$$

So

$$H = P^T \dot{Q} - \overbrace{\left( \frac{1}{2} \dot{Q}^T A \dot{Q} + B^T \dot{Q} - U \right)}^L$$

$$= (P - B)^T \dot{Q} - \left( \frac{1}{2} \dot{Q}^T A \dot{Q} - U \right)$$

$$= \frac{1}{2} (P - B)^T A^{-1} (P - B) + U$$

use Eq  $\star$