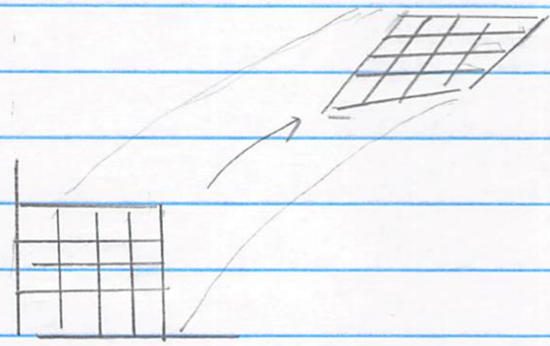
Brief Review in 1D

- The Hamiltonian evolultion can be thought of a map in phase space



$$(q, p) \to (Q, P)$$

- The map preserves the volume of phase space

- Passive view: This is just a new set of coordinates for the phase space

$$Q(q, p) \qquad P(q, p)$$

- Active view: The system evolves in the phase space:

$$q \to Q = q + \dot{Q}(q,p)\, t \qquad \dot{Q} = dQ/dt$$

$$p \to P = p + \dot{P}(q,p)\, t \qquad \dot{P} = dP/dt$$

- This evolution preserves the phase-space volume

$$\boxed{\frac{d(\Delta Q \Delta P)}{dt} = \left(\frac{\partial \dot{Q}}{\partial q} + \frac{\partial \dot{P}}{\partial p} \right) \Delta p \Delta q}$$

Canonical Transformation Review

A. Then we asked are there other "canonical" transformations which leave the form of Hamilton's Equations unchanged

$$\tilde{H}(P,Q) = H(p,q) \qquad \begin{aligned} q &\rightarrow Q(q,p) \\ p &\rightarrow P(q,p) \\ H &\rightarrow \tilde{H}(P,Q) \end{aligned} \qquad \begin{aligned} \dot{Q} &= \frac{\partial \tilde{H}}{\partial P} \\ \dot{P} &= -\partial \tilde{H}/\partial Q \end{aligned}$$

B. We considered infinitessimal transformations depending on a parameter λ. The only allowed infinitetessimal canonical transformations take the form

$$\begin{aligned} q^i &\rightarrow Q^i = q^i + \partial G/\partial p_i \, \lambda \\ p_i &\rightarrow P_i = p_i - \partial G/\partial q^i \, \lambda \end{aligned} \qquad \begin{aligned} dQ/d\lambda &= \partial G/\partial p \\ dP/d\lambda &= -\partial G/\partial q \end{aligned}$$

For some function G, called the generator of transformation.

C. The transformation preserves the Symplectic form. This means that if we write:

$$z^i = (q, p)$$

So that

$$\dot{z}^i = J^{ij} \frac{\partial H}{\partial z^j} \qquad \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array} \right) = J^{ij}$$

symplectic matrix

Then if the canonical transform is:

$$z \to y \qquad \text{with} \qquad M^i_j = \frac{\partial z^i}{\partial y^j}$$

$$H(z) = \tilde{H}(y)$$

Jacobian map

then

$$\boxed{M J M^T = J.}$$

with

$$\dot{y}^i = J^{ij} \frac{\partial \tilde{H}}{\partial y^j}$$

D. Finally Phase Space is conserved:

$$\frac{d}{d\lambda}(\Delta Q \Delta P) = \left(\frac{\partial (dQ/d\lambda)}{\partial q} + \frac{\partial (dP/d\lambda)}{dp} \right) \cdot \Delta p \Delta q$$

Which follows immediately the fact $G(q,p)$ generates the transformation

$$\frac{\partial^2 G}{\partial q \partial p} - \frac{\partial^2 G}{\partial p \partial q} = 0$$

E. Change under G of observable $O(p,q)$:

Then $\delta O \equiv O(P,Q) - O(p,q)$ is

$$\delta O = \frac{\partial O}{\partial q}\,\delta q + \frac{\partial O}{\partial p}\,\delta p$$

So $\delta q = +\partial G/\partial p\,\lambda$, etc so

$$\delta O = \left(\frac{\partial O}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial O}{\partial p}\frac{\partial G}{\partial q}\right)\lambda$$

$$\boxed{\delta O = \{O, G\}\,\lambda}$$

← transformation of O under G

- So if G leaves H invariant $\delta H = 0$ then

$$\boxed{\mathring{G} = \{G, H\} = 0 \leftarrow G \text{ is constant in time}}$$

Important Examples of Generators

Translations:

- $q \longrightarrow q + \lambda \qquad p \to p$
- You can easily check that this transformation is generated by

$$G(q,p) = p$$

- More generally with canonical variables (q^i, p_i) the function

$$G = a^i p_i$$

generates the transformation

$$q^i \to q^i + \lambda a^i \qquad p_i \to p_i$$

Rotations:

Consider a rotation in the phase space (x, y, p_x, p_y)

$$x(\theta) = x\cos\theta + y\sin\theta$$

$$y(\theta) = -x\sin\theta + y\cos\theta$$

$$p_x(\theta) = p_x\cos\theta + p_y\sin\theta$$

$$p_y(\theta) = -p_x\sin\theta + p_y\cos\theta$$

- So for small θ

$$x \to x + y\,\delta\theta \qquad p_x \to p_x + p_y\,\delta\theta$$
$$y \to y - x\,\delta\theta \qquad p_y \to p_y - p_x\,\delta\theta$$

- Then this is generated by

$$G = y p_x - x p_y$$ (Check me!)

- If more generally

$$G = n \cdot \vec{L} = \vec{n}\cdot(\vec{r}\times\vec{p})$$

Then

$$\delta\vec{r} = \delta\theta \frac{\partial G}{\partial\vec{p}} = \delta\theta \frac{\partial}{\partial\vec{p}}\, \vec{p}\cdot(\vec{n}\times\vec{r})$$

$$\delta\vec{r} = \delta\theta\, \vec{n}\times\vec{r}$$

Similarly

$$\delta\vec{p} = -\delta\theta\, \partial G/\partial\vec{r} = -\delta\theta \frac{\partial}{\partial\vec{r}}\, \vec{r}\cdot(p\times\vec{n}) = \delta\theta\, \vec{n}\times\vec{p}$$

Finally we may take the Hamiltonian itsself

$$q \to q + \frac{\partial H}{\partial p} t \qquad \dot{q} = \frac{\partial H}{\partial p}$$

$$p \to p - \frac{\partial H}{\partial q} t \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

So we that the Hamiltonian generates time translations

Poisson Bracket and Canonical Transformations

- Poisson Brackets are invariant under canonical transformations. For a direct proof using the Symplectic matrix see Tong.
- Take an observable $w(p,q)$ (e.g. kinetic energy) In the new coordinates the observable has a different functional form, but same value:

$$\tilde{W}(P,Q) = w(p,q) \quad \text{at all time}$$

But

$$\frac{d\tilde{W}}{dt} = \{\tilde{W}, \tilde{H}\}_{PQ} = \frac{dW}{dt} = \{w, H\}_{pq}$$

So these $\{\tilde{W}, \tilde{H}\}_{PQ} = \{w, H\}_{pq}$ are the same

for all w and H

- But nowhere did we use the specific form of H, or $\tilde{H}$, or w or W
- So this is a relation between Poisson brackets

$$\boxed{\{\tilde{F}, \tilde{G}\}_{PQ} = \{f, g\}_{pq}},$$

(where

$$\tilde{F}(P, Q) = f(p, q),$$

have the same values but different functional forms)

- Thus we are led to a new statement about canonical transforms. If a transform preserves the Poisson structure

$$\boxed{\begin{gathered}\{Q_i, Q_j\}_{p,q} = \{P_i, P_j\}_{p,q} = 0 \\ \{Q_i, P_j\}_{pq} = \delta_{ij}\end{gathered}}$$

Then the transform

$$q \to Q(q, p)$$

$$p \to P(q, p)$$

is canonical