The Hamiltonian Function and energy

- Define:

$$h(q,\dot{q},t) \equiv p\,\dot{q} - L(q,\dot{q},t) \qquad p \equiv \frac{\partial L}{\partial \dot{q}}(q,\dot{q},t)$$

The Hamiltonian function

- we use a lower case h, because we consider this to be a function of $q, \dot{q}$. Now:

$$\frac{dh}{dt} = \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} + \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} - \frac{\partial L}{\partial q}\dot{q}}_{\text{this vanishes by the equation of motion}} - \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} - \frac{\partial L}{\partial t}$$

- So

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}$$

← Thus if the Lagrangian is not an explicit function of time then $h(q,\dot{q},t)$ will be constant

- For a lagrangian of the form

$$L = \frac{1}{2}a_{ij}(q)\,\dot{q}^i\dot{q}^j + b_i(q)\,\dot{q}^i - V(q,t)$$

symmetric

Then

$$p_i = a_{ij}\dot{q}^j + b_i = \frac{\partial L}{\partial \dot{q}_i}$$

- And

$$h = p_i \dot{q}^i - L$$

$$= (a_{ij}\dot{q}^j + b_i)\dot{q}^i - \left(\frac{1}{2} a_{ij}\dot{q}^i \dot{q}^j + b_i \dot{q}^i - V\right)$$

$b_i \dot{q}^i$ terms cancel

$$\boxed{h = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + V}$$

- So the kinetic energy is

$$T = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j$$

And

$$h = T + V$$

This conserved quantity is known as the first integral

- This is very relevant for a particle in a magnetic field