Phase Space

- It is informative to study the motion of the system in the x-p plane. Take the simple pendulum:

$\theta \simeq \pi$

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = m l^2 \dot\theta$$

- For small oscillations, a trajectories energy is:

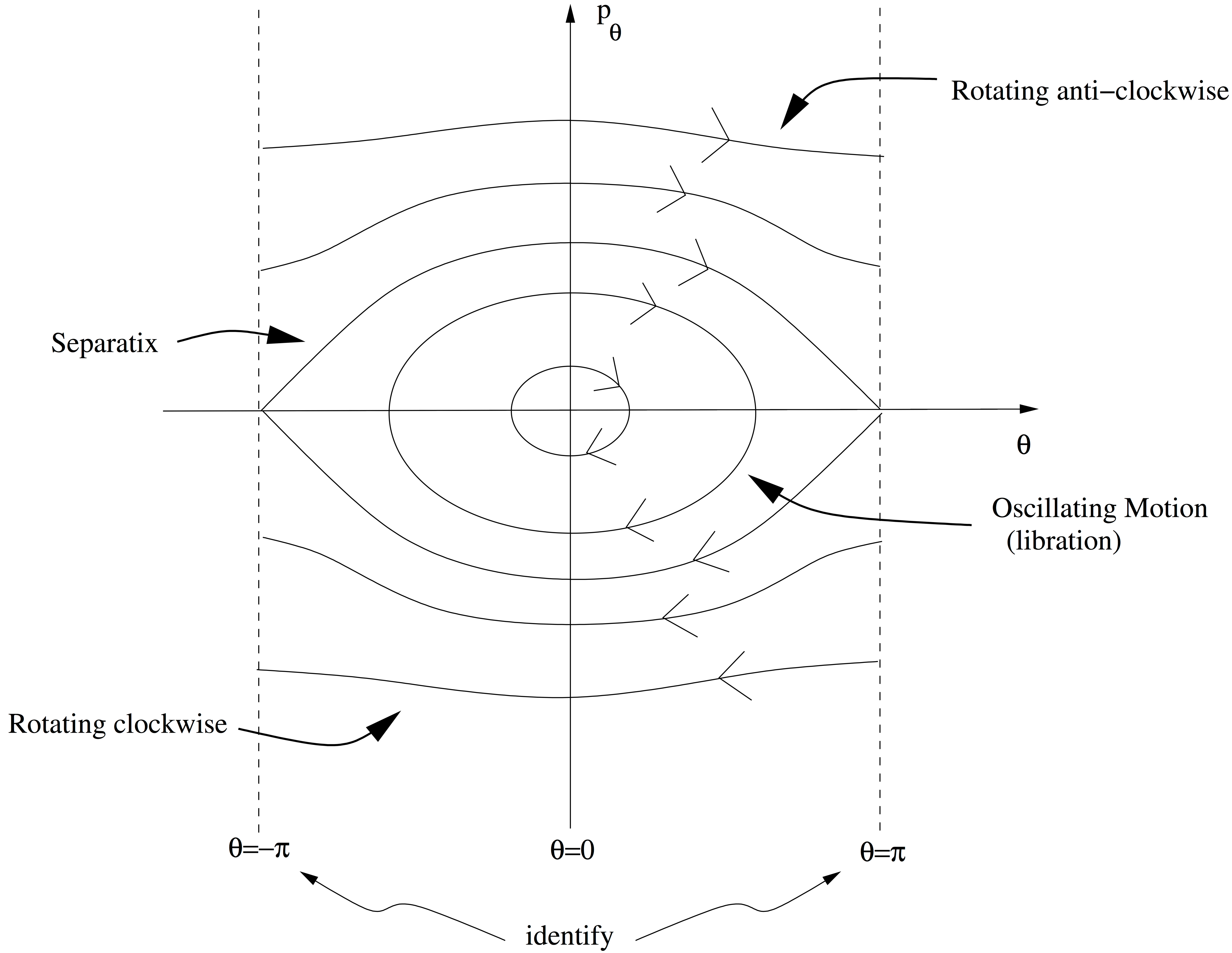
$$E = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

$$\simeq \frac{p_\theta^2}{2ml^2} + mgl\frac{\theta^2}{2} + \text{const}$$

- Thus in the phase space (θ, p_θ) the trajectory is a circle, after choosing appropriatie units $m = g = l = 1$.

- For larger values of E, the circles are deformed, and ultimately changed completely when $E > E_{separatrix} = 2mgl$. Then the pedulum goes around in a circle, clockwise or counter-clockwise
- We see how imformative the phase space can be!

Phase portrait of a pendulum



Liouville Theorem

- Now consider another phase-space plot. (next page) For an ensemble of particles undergoing constant acceleration ($x(t) = x_0 + V_0 t + 1/2 a t^2$)

$$\left.\begin{aligned} Q(t) &= \frac{1}{2} a t^2 + q + \frac{p}{m} t \\ P(t) &= m a t + p \end{aligned}\right\}$$

A map from:

$$(q, p) \longrightarrow (Q, P)$$

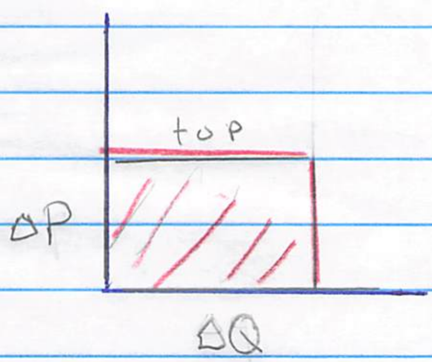
initial state — final state

- The next figure shows how an ensemble of particles starting in the square in the lower left evolves in time (see next page)

Liouville Theorem

- Quite generally the "phase-space volume" covered by the ensemble, does not grow in magnitude in time, but stretches and changes shape:

Simple Proof



$$q \rightarrow Q(t) = q + \dot{Q}(p,q)\, t$$

$$p \rightarrow P(t) = p + \dot{P}(q,p)\, t$$

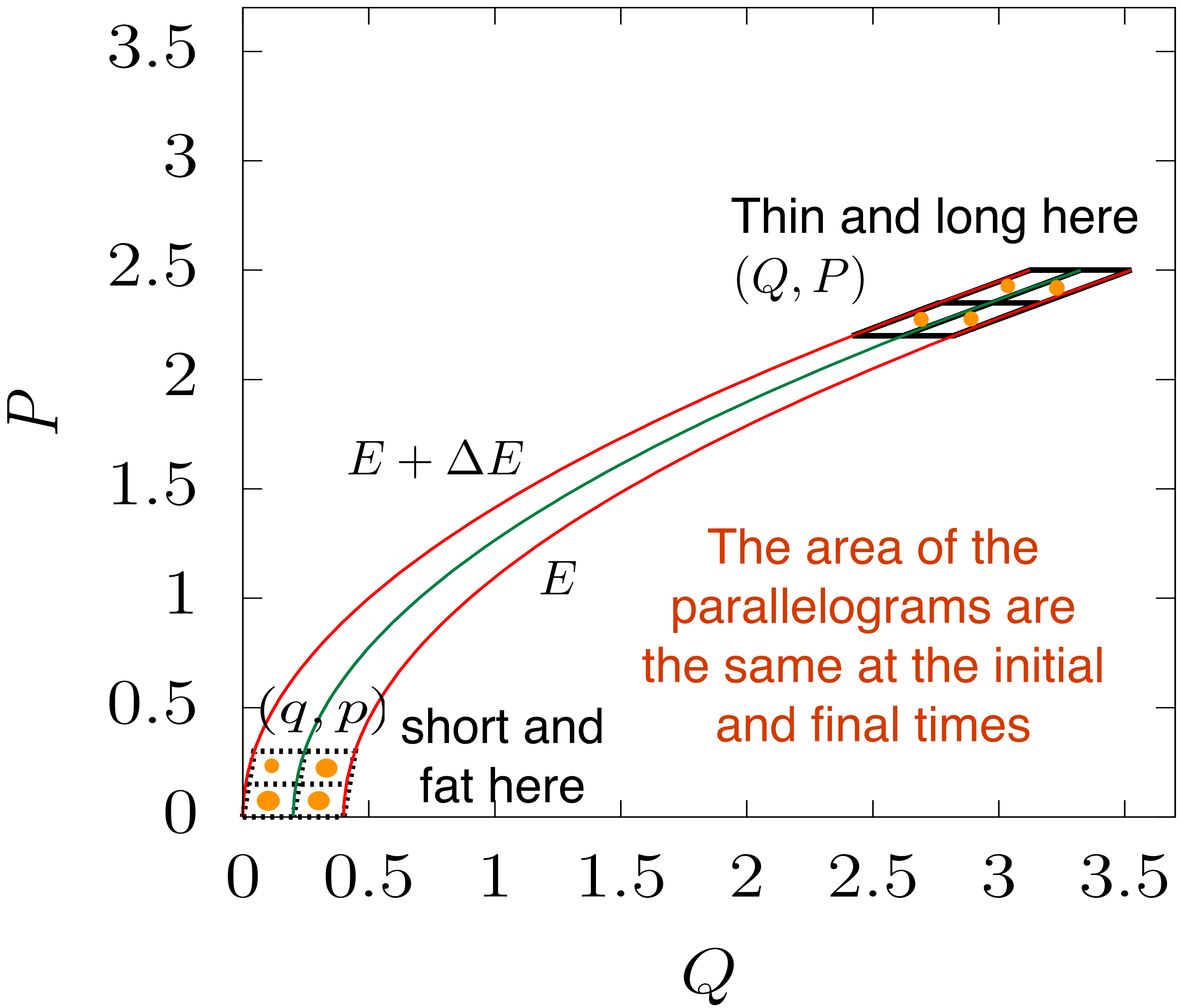
t is small

$$\dot{Q}(q,p) = \frac{\partial H(q,p)}{\partial p} \qquad \dot{P} = \frac{\partial H(q,p)}{\partial q}$$

- How does the volume change in time?

$$d(\Delta P) = \left(\dot{P}\Big|_{top} - \dot{P}\Big|_{bottom} \right) dt = \frac{\partial \dot{P}}{\partial p} \Delta p \, dt$$

Phase space volume conserved



Similarly

$$d(\Delta Q) = \left(\dot{Q}\Big|_{\text{right}} - \dot{Q}\Big|_{\text{left}} \right) dt$$

$$= \frac{\partial \dot{Q}}{\partial q} \Delta q \, dt$$

- So

$$d(\Delta P \Delta Q) = \Delta P \, d(\Delta Q) + d(\Delta P) \Delta Q$$

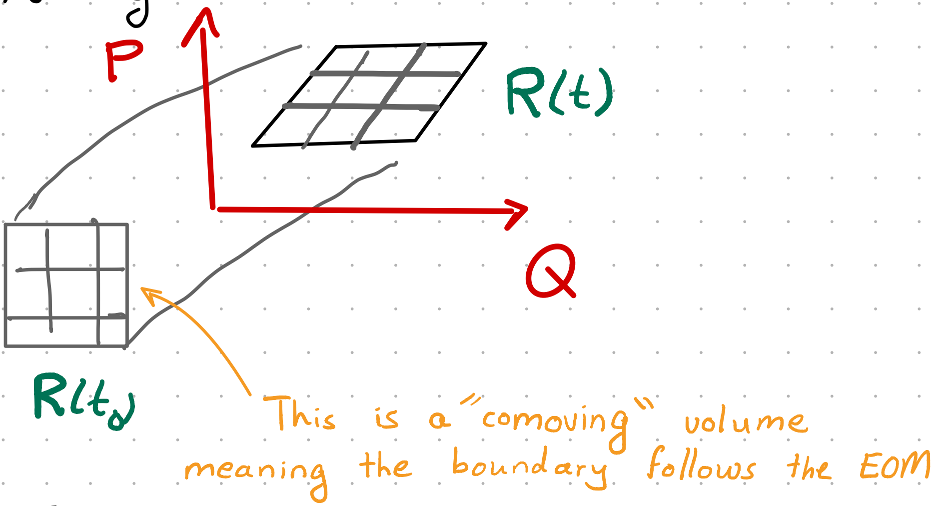
$$= \left(\frac{\partial \dot{Q}}{\partial q} + \frac{\partial \dot{P}}{\partial p} \right) dt \, \Delta p \, \Delta q$$

So

$$\boxed{d(\Delta P \Delta Q) = \left(\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q} \right) dt \, \Delta p \, \Delta q = 0}$$

Change in phase-space volume of square

- A more formal procedure/proof is the following:



- Then the area at time t is

$$A(t) = \int_{R(t)} dQ\,dP = \int_{R(t_0)} \left|\frac{\partial(Q,P)}{\partial(q,p)}\right| dq\,dp$$

we will show now that the determinant is unity. Thus the areas at the final and inital times are equal

$$A(t) = \int_{R(t_0)} dq\,dp = A(t_0)$$

Proof:

$$Q(t) = q + \frac{\partial H}{\partial p}(q,p)\,t$$

$$P(t) = p - \frac{\partial H}{\partial q}(q,p)\,t$$

So

$$\left|\frac{\partial(Q,P)}{\partial(q,p)}\right| = \begin{vmatrix} \partial Q/\partial q & \partial Q/\partial p \\ \partial P/\partial q & \partial P/\partial q \end{vmatrix}$$

$$= \begin{vmatrix} 1 + t\dfrac{\partial^2 H}{\partial q \partial p} & t\dfrac{\partial^2 H}{\partial p^2} \\ t\,\partial^2 H/\partial p^2 & 1 - t\,\partial^2 H/\partial p\,\partial q \end{vmatrix}$$

$$= 1 + t\left(\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q}\right) + \underbrace{O(t^2)}_{\text{neglect}}$$

$$= 1 + O(t^2)$$

So the volume is preserved by the map.

More Coordinates

In general

$$A(t) = \int_{R(t)} dQ^1 dP_1\, dQ^2 dP_2 \cdots$$

$$= \int_{R(t_0)} \left|\frac{\partial(Q,P)}{\partial(q,p)}\right| dq^1 dp_1\, dq^2 dp_2 \cdots$$

$$= A(t_0),$$

provided this determinant is unity (it is!)

Proof:

$$q^i \to Q^i = q^i + \frac{\partial H}{\partial p_i} t$$

$$p_i \to P_i = p_i - \frac{\partial H}{\partial q_i} t$$

The Jacobian of the map is

$$M = \det \begin{pmatrix} \frac{\partial Q^i}{\partial q^j} & \frac{\partial Q^i}{\partial p_j} \\ \frac{\partial P_i}{\partial q^j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix}$$

$$= \det \begin{pmatrix} \delta^i_j + \frac{\partial^2 H}{\partial p_i \partial q^j} t & \frac{\partial^2 H}{\partial p_i \partial p_j} t \\ t \frac{\partial^2 H}{\partial q^i \partial q^j} & \delta^j_i - t \frac{\partial^2 H}{\partial p_j \partial q^i} \end{pmatrix}$$

- Then note for a matrix M

$$\det M = \exp \operatorname{Tr} \log M$$

just prove me by working in the eigenbasis of M

- Now in this case the matrix M is close to the identity

small time

$$M = \mathbb{1} + t\, M^{(1)}$$

$$\log M \simeq t\, M^{(1)} \,, \quad \mathrm{Tr} \log M \simeq t\, \mathrm{Tr}\, M^{(1)}$$

$$\det M \simeq e^{t\, \mathrm{Tr}\, M^{(1)}} \simeq 1 + t\, \mathrm{Tr}\, M^{(1)}$$

- But in this case

$$\mathrm{Tr}\, M^{(1)} = \frac{\partial^2 H}{\partial p_i \partial q^i} - \frac{\partial^2 H}{\partial p_i \partial q^i} = 0$$

So the volume in phase space of any comoving region is constant in time!

(since the det of the map is unity)