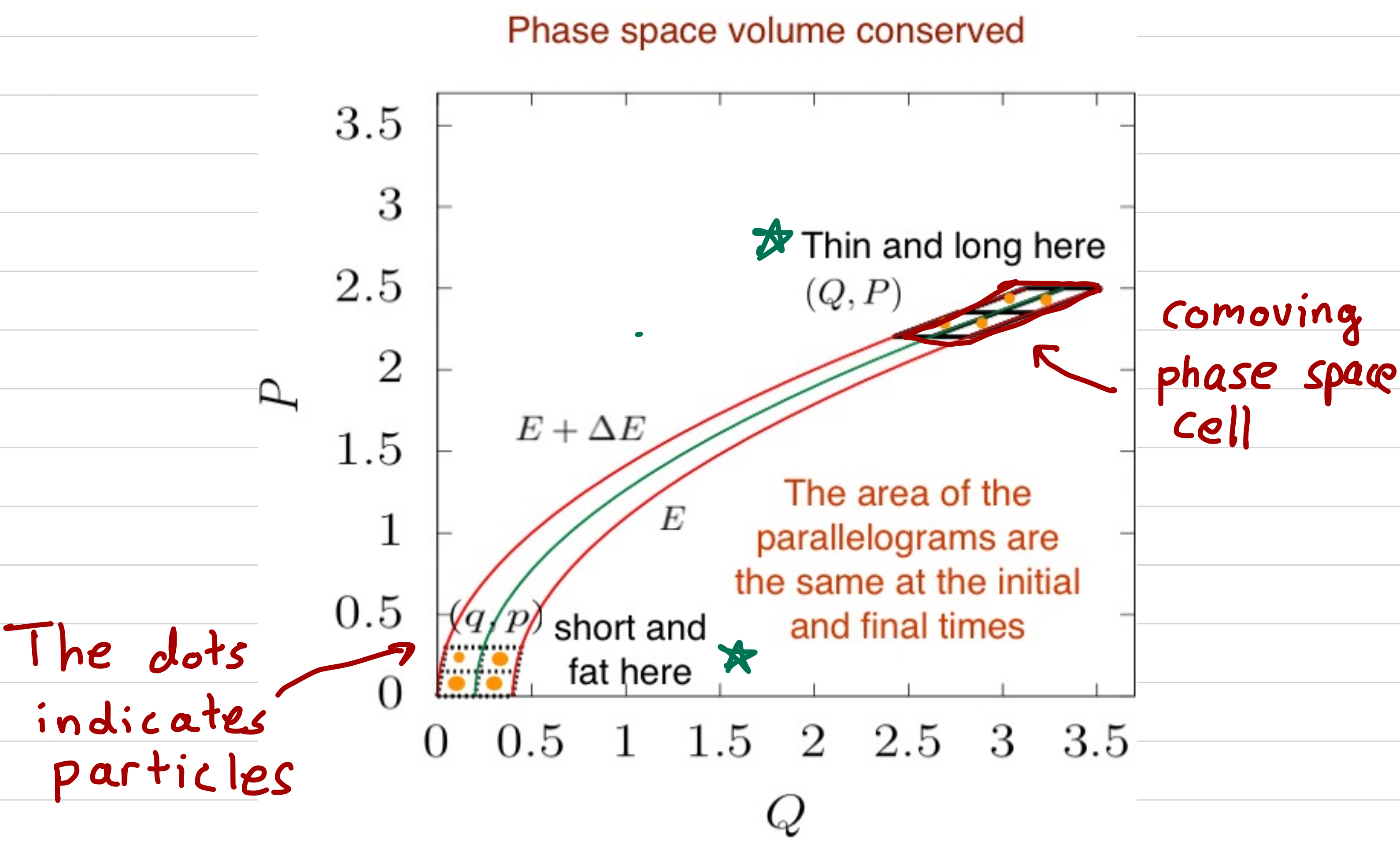
Phase space density

- We have discussed that the phase space volume of any given "comoving" region remains fixed in time.

- By co-moving we mean that the boundaries of the region evolve with the EOM. (see below)



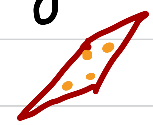
- The phase space density is

$$f(t, q, p) = \frac{dN}{dq\, dp}$$ is the

number of particles per phase space volume.

- The number of partices is

$$N = \int dq\, dp\; f(t, q, p)$$

- The number of particles in each "comoving" cell  can not change under the Hamiltonian evolution. Since, if a particle touches the boundary, it has the same initial condition (q, p) for the subsequent evolution as the boundary contact point. The boundary point and the particle would then follow each other thereafter. In addition, if we run the Hamiltonian evolution in reverse, then the particle would remain on the boundary at all negative times.

- Since the number in a cell is constant and the area of the cell is constant, the phase space density $f(t, q, p)$ is also constant in time.

$$\boxed{\frac{df}{dt} = 0}$$

Now

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\frac{dq}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} = 0}$$

So since $\dot{x} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial x$ we have

$$\boxed{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q} = 0}$$

Liouville Equation

The structure is known as the Poisson bracket

$$\boxed{\{f, H\} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}}$$

we will discuss Poisson Brackets in more generality in the next section. For now we note that the Liouville equation can be written

$$\boxed{\frac{\partial f}{\partial t} + \{f, H\} = 0}$$