Symplectic Integrators

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Lectures

- Consider a straightforward discretization of the Hamiltonian Equations of motion

Explicit Euler Method

$$\delta q^{n+1} = q^n + \delta t \frac{\partial H}{\partial p}(q^n, p^n)$$

$$\delta p^{n+1} = p^n - \delta t \frac{\partial H}{\partial q}(q^n, p^n)$$

- Regard this as a map from $q^n, p^n \to (q^{n+1}, p^{n+1})$ this map is not canonical for finite δt since

$$\frac{\partial(q^{n+1}, p^{n+1})}{\partial(q^n, p^n)} = \begin{vmatrix} 1 + \delta t \dfrac{\partial H}{\partial q \partial p} & \delta t \dfrac{\partial H}{\partial p^2} \\ \delta t \dfrac{\partial H}{\partial q^2} & 1 - \dfrac{\partial^2 H}{\partial q \partial p} \end{vmatrix}$$

$$\neq 1 \quad \text{for finite } \delta t$$

- However we have a general scheme to construct maps which are canonical. The Jacobian associated with these maps will be unity

Consider the Generator

$$\Phi(q, P) = qP + \delta t\, H(q, P)$$

For small δt this is close to the identity, but we will consider δt finite. It generates the map

$$\star \quad Q = \frac{\partial \Phi}{\partial P} = q + \delta t\, \frac{\partial H}{\partial P}(q, P)$$

$$\star\star \quad p = \frac{\partial \Phi}{\partial q} = P + \delta t\, \frac{\partial H}{\partial q}(q, P)$$

take for definiteness $H = P^2/2m + U(q)$ then

$\star\star$ gives with $q^{n+1} = Q$ and $p^{n+1} = P$

the symplectic Euler method

$$p^{n+1} = p^n - \delta t\, \frac{\partial U}{\partial q}(q^n)$$

$$q^{n+1} = q^n + \delta t\, \frac{p^{n+1}}{m}$$

notice we update p first, and then update q

This map does preserve the phase space volume:

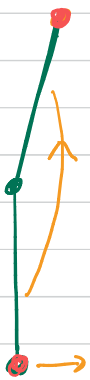
$$\frac{\partial(q^{n+1}\ p^{n+1})}{\partial(q^n\ p^n)} = \begin{vmatrix} 1 & 0 \\ 1 - \delta t\, \frac{\partial^2 U}{\partial q^2}(q^n) & 1 \end{vmatrix} = 1 \ !$$

For integration over long times preserving the phase space volume is essential for the qualitative features of Hamiltonian evolution

The following shows the phase portrait of the the pendulum and the flow of phase space:

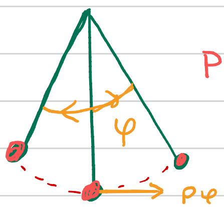
$$H = \frac{p_\varphi^2}{2ml^2} - mgl\cos\varphi \qquad p_\varphi = ml^2\dot{\varphi}$$

A

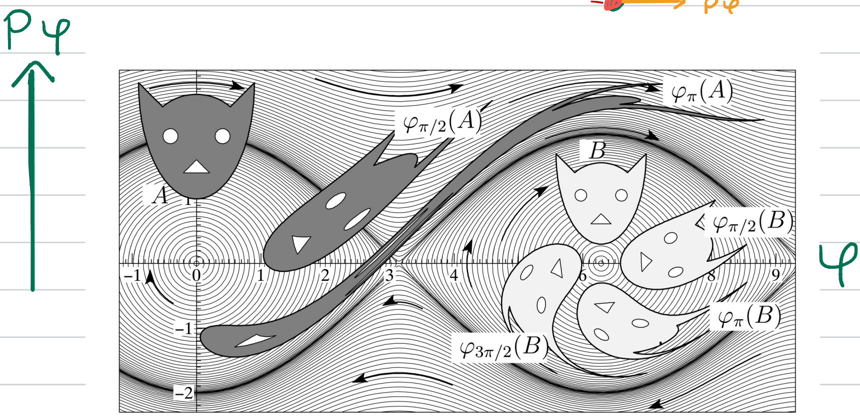


If moving fast enough, it circles around rather than oscillating. The cat crosses the dividing line (separatrix)

B



phase space flow for oscillations



$m = g = l = 1$

The flow preserves area!

The following shows how the maps work for $\delta t = \pi/4$. The symplectic map preserves area and is much preferred for long time integration in stat. mech and celestial mechanics.

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