Problem 1. (MIT/OCW) Spring system on a plane

A massless spring has an unstretched length b and spring constant k, and is used to connect two particles of mass m_1 and m_2. The system rests on a frictionless table and may oscillate, translate, and rotate.

(a) What is the Lagrangian? Write it with two-dimensional cartesian coordinates $\boldsymbol{r}_1 = (x_1, y_1)$ and $\boldsymbol{r}_2 = (x_2, y_2)$. There are four coordinates in total.

(b) Setup a suitable set of generalized coordinates (four in total) to better account for the symmetries of this system. Take one of your coordinates to be $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. What is the Lagrangian in these variables?

(c) Identify three conserved generalized momenta that are associated to cyclic coordinates in the Lagrangian from part (b). If you think you are missing some, try to improve your answer to (b). Briefly explain the physical meaning of each of the three conserved generalized momenta. Show that the equation of motion for r takes the form

$$m_{\text{eff}}\ddot{r} = -\frac{\partial V_{\text{eff}}(r)}{\partial r} \tag{1}$$

with an appropriate m_{eff} and $V_{\text{eff}}(r)$.

(d) Write down the hamiltonian function $h(q, \dot{q}, t)$ for the coordinates chosen in (b). Show that that the velocity $\dot{r}$ associated with the coordinate r (here $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ is distance between the particles) can be determined from the energy E and an effective potential $V_{\text{eff}}(r)$ which depends on the rotation rate, i.e. show that

$$\frac{1}{2}m_{\text{eff}}\dot{r}^2 + V_{\text{eff}}(r) = E \tag{2}$$

(e) By examining the effective potential and its dependence on the rotation rate, show that there is a solution that rotates but does not oscillate, and discuss what happens to this solution for an increased rate of rotation. (A closed form solution is not necessary. A graphical explanation based on the effective potential will suffice.)

Problem 2. (Goldstein/MIT OCW) Jerky Mechanics

Consider an extension of classical mechanics where the equation of motion involves a triple time derivative, $\dddot{x} = f(x, \dot{x}, \ddot{x}, t)$. Lets use the action principle to derive the corresponding Euler-Lagrange equations. Start with a Lagrangian of the form $L(q^i, \dot{q}^i, \ddot{q}^i, t)$ for n generalized coordinates q^i, and make use of the action principle for paths $q^i(t)$ that have zero variation of both q^i and $\dot{q}^i$ at the end points. Show that

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}^i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) + \frac{\partial L}{\partial q^i} = 0 \tag{3}$$

for each $i = 1 \ldots n$

Problem 3. Equivalent Lagrangians

(a) (Goldstein) Let $L(q, \dot{q}, t)$ be the Lagrangian for a particle with coordinate q, which satisfies the Euler-Lagrange equations. Show that the Lagrangian

$$L' = L + \frac{dF(q,t)}{dt} \tag{4}$$

yields the same Euler-Lagrange equations as L where F is an arbitrary differentiable function. Give a proof based on and the action principle. We say that L and L' are equivalent. (If you feel like it you might also like to check directly that the EOM are the same.)

(b) (Goldstein) Using the previous problem (Problem 3), what is the equation of motion resulting from

$$L = -\frac{1}{2}mq\ddot{q} - \frac{1}{2}\omega_0^2 q^2 \tag{5}$$

and what is it related to? Explain why this equation of motion is obvious from the Lagrangian in Eq. (5) and the result of part (a).

(c) Consider the action of a free particle

$$S[\boldsymbol{r}(t)] = \int \mathrm{d}t\, C\boldsymbol{v}^2 \tag{6}$$

where $C = m/2$ is a constant normally associated with the mass. Show that the action is unchanged (up to boundary terms) by a Galilean transformation, and hence the transformed version gives the same EOM. If the Lagrangian took the form $L = Cv^4$ this would not have been the case. Thus requiring Gallilean invariance fixes the form the velocity dependent action to involve the kinetic energy.

(d) Consider a fricitionless block of mass m in one dimension. The block sits on a train, which accelerates with constant acceleration a_0. The block experiences no forces, and thus the action of the block is simply the free one

$$S = \int \mathrm{d}t \frac{1}{2}mv_g^2\,, \tag{7}$$

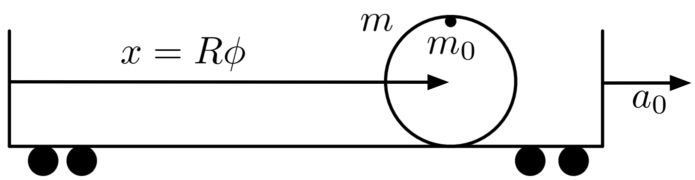
where $v_g(t)$ is the velocity relative to the ground. Let $v(t)$ denote the velocity of the block relative to the back of the train.

(i) Write down the relation between $v(t)$ and $v_g(t)$, and substitute into Eq. (7) to determine the Lagranngian for $v(t)$.

(ii) Show that this Lagrangian is equivalent to that of a particle in a potential $U(x) = ma_0 x$ where x is the position of the particle relative to the back of the train, and interpret the result.

Problem 4. A cylinder on a train

Consider a cylinder-like contraption consisting of a cylindrical ring of mass m and radius R, and a small weight of mass m_0 fixed to the rim of the ring (see below). At time $t = 0$ the cylinder starts to roll without slipping from rest in the accelerating train, and the weight is at the top of its arc as shown in the figure below.



(a) Determine the Lagrangian for the angle $\phi(t)$. Here $x \equiv R\phi$ is the position of the center of the cylinder relative to the back of the train (see figure). Show that the Lagrangian for ϕ may be written in a time independent form

$$L = \frac{1}{2} m_{\rm eff}(\phi)\, R^2\, \dot{\phi}^2 - U(\phi)\,, \tag{8}$$

where $m_{\rm eff}(\phi)$ and $U(\phi)$ are specific functions of ϕ

$$m_{\rm eff}(\phi) = 2m + 2m_0(1 + \cos\phi) \tag{9}$$

$$U(\phi) = (m + m_0)a_0\, R\phi + m_0 a_0 R \sin(\phi) + m_0 g \cos\phi \tag{10}$$

Hint: As in part (d) of the last problem, in the train's frame the acceleration acts like an additional gravitational field of magnitude a_0 pulling in the negative x direction.

(b) What is the speed of the cylinder after it rolls for two complete turns.

Hint: use the first integral $h(q, \dot{q}, t)$ associated with the Lagrangian in Eq. (8).